Ma3 Ch2 Corrigé des exercices

ex 1

a) $\frac{f(2)-f(1)}{2-1} = \frac{2^2-1^2}{2-1} = \frac{3}{1} = 3$

b) $\frac{f(1{,}5)-f(1)}{1{,}5-1} = \frac{1{,}5^2-1^2}{0{,}5} = \frac{2{,}25-1}{0{,}5} = \frac{1{,}25}{0{,}5} = 2{,}5$

c) $\frac{f(2)-f(-2)}{2-(-2)} = \frac{2^2-(-2)^2}{2+2} = \frac{4-4}{4} = 0$

d) $\frac{f(2)-f(1)}{2-1} = \frac{2^3-1^3}{1} = 7$

ex 2

a) $f'(1) = \lim\limits_{x\to1} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to1} \frac{x^2-1^2}{x-1}$ type "$\frac{0}{0}$"

$= \lim\limits_{x\to1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 1+1 = 2$

$f'(3) = \lim\limits_{x\to3} \frac{x^2-3^2}{x-3} = \lim\limits_{x\to3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = 3+3 = 6$

b) $f'(2) = \lim\limits_{x\to2} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to2} \frac{x^3-2^3}{x-2} = \lim\limits_{x\to2} \frac{x^3-8}{x-2}$

$= \lim\limits_{x\to2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{x-2}} = 2^2+2\cdot2+4 = 12$

c) $f'(-2) = \lim\limits_{x\to-2} \frac{f(x)-f(-2)}{x-(-2)} = \lim\limits_{x\to-2} \frac{x-(-2)}{x+2} = \lim\limits_{x\to-2} \frac{x+2}{x+2}$

$= \lim\limits_{x\to-2} \frac{1}{1} = 1$

$f'(5) = \lim\limits_{x\to2} \frac{f(x)-f(5)}{x-5} = \lim\limits_{x\to5} \frac{\cancel{x-5}^{\,1}}{\cancel{x-5}_{\,1}} = \lim\limits_{x\to5} 1 = 1$

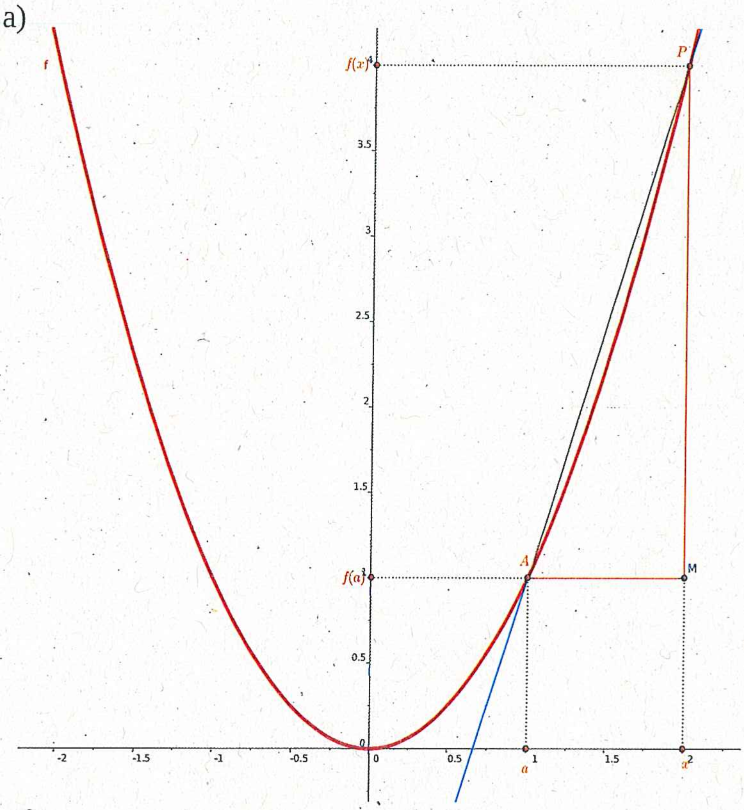
d) $f'(2) = \lim\limits_{x\to2} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x\to2} \frac{3-3}{x-2} = \lim\limits_{x\to2} \frac{0}{x-2} = \lim\limits_{x\to2} 0 = 0$

$f'(7) = \lim\limits_{x\to2} \frac{f(x)-f(7)}{x-7} = \lim\limits_{x\to2} \frac{3-3}{x-7} = \lim\limits_{x\to2} \frac{0}{x-7} = \lim\limits_{x\to2} 0 = 0$

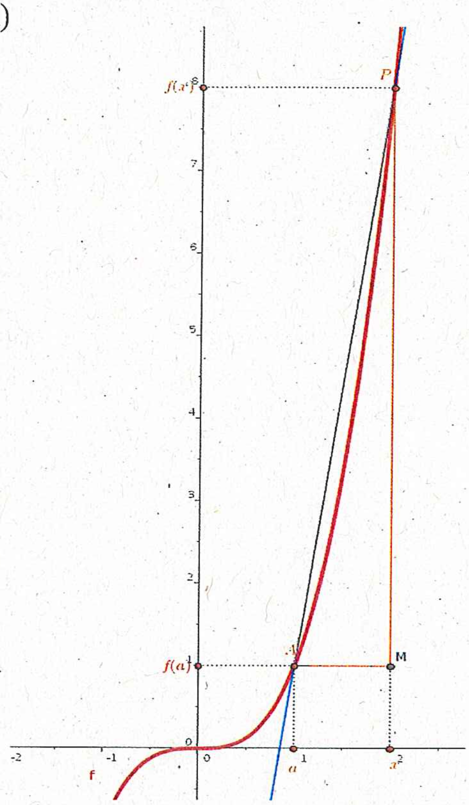
Interprétations graphiques :

ex1

a)

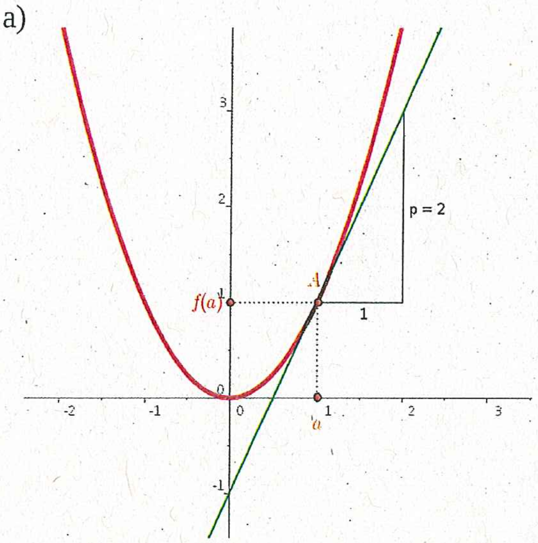


d)



ex2

a)



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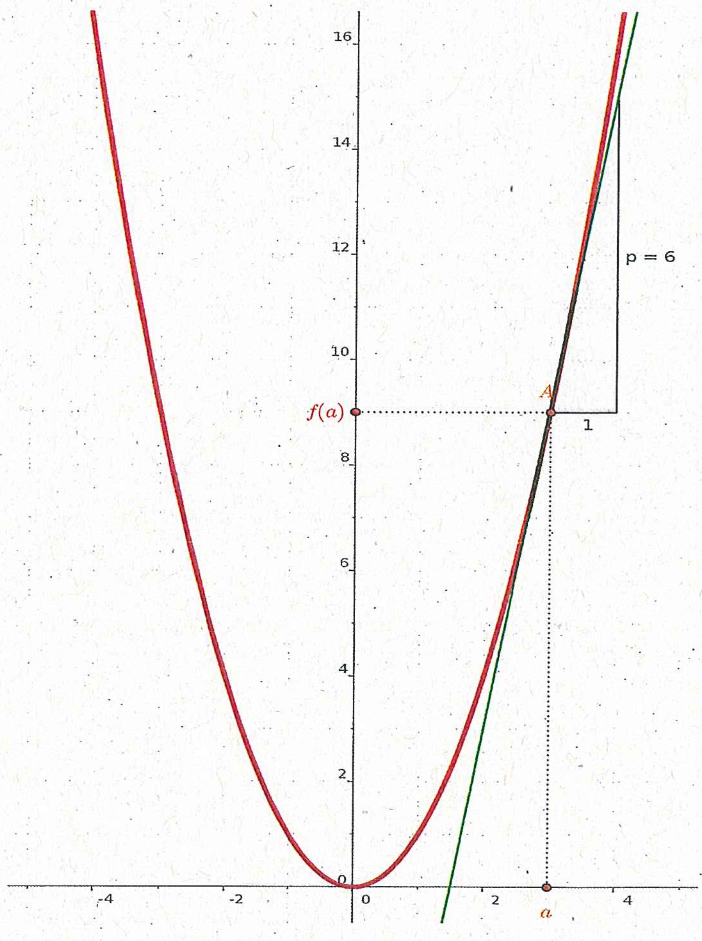
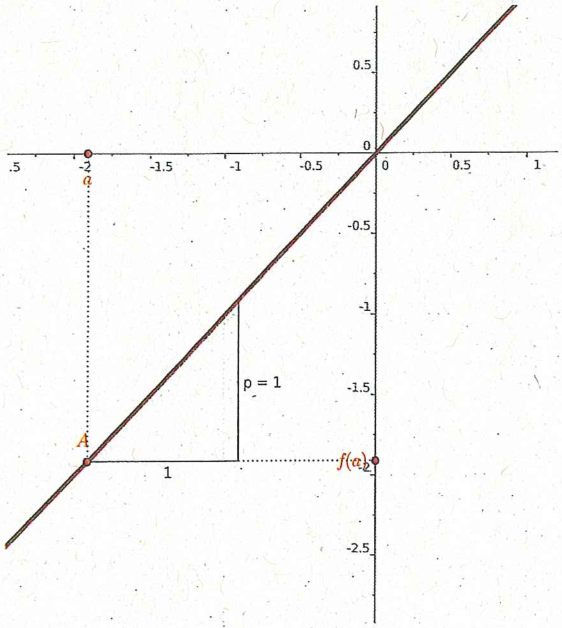


c)



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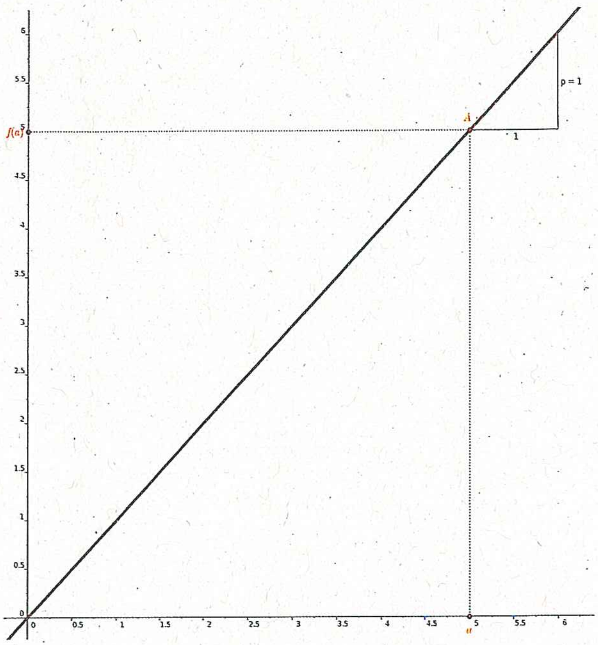
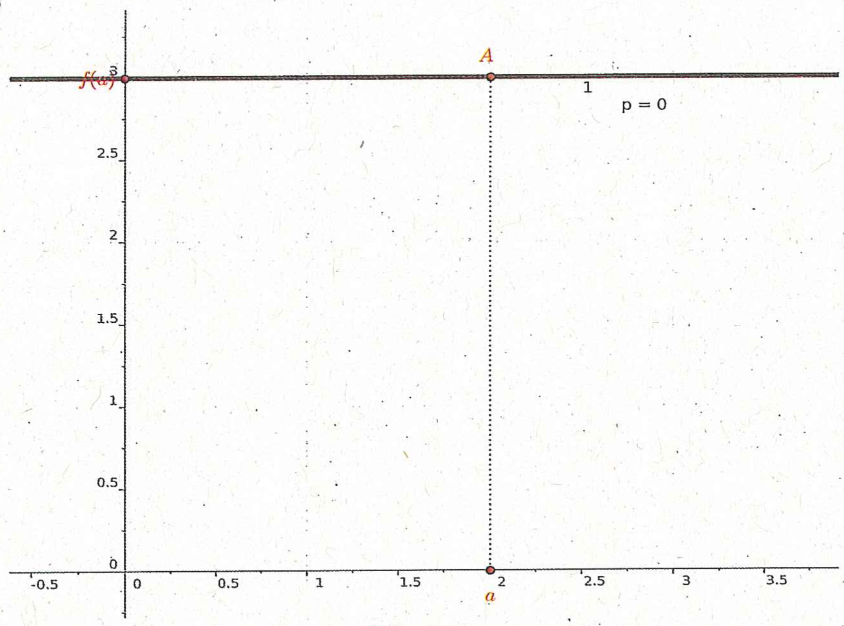


d)



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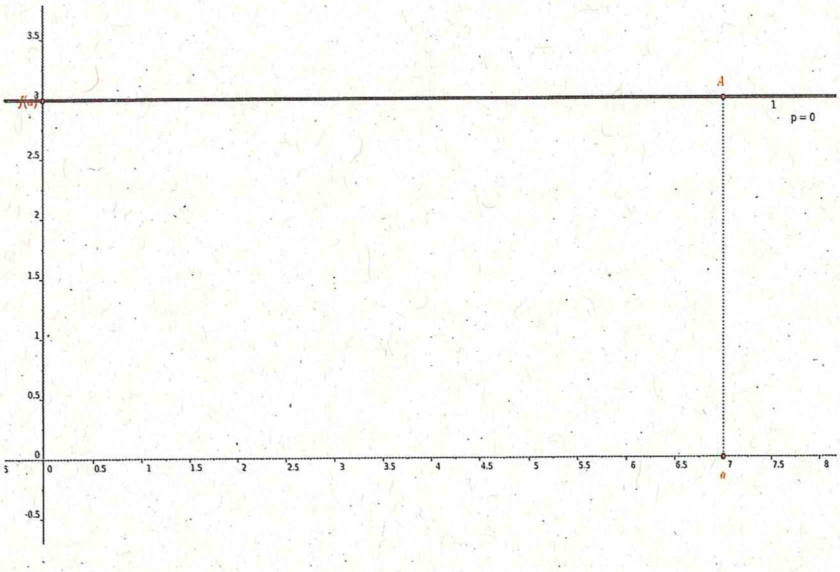
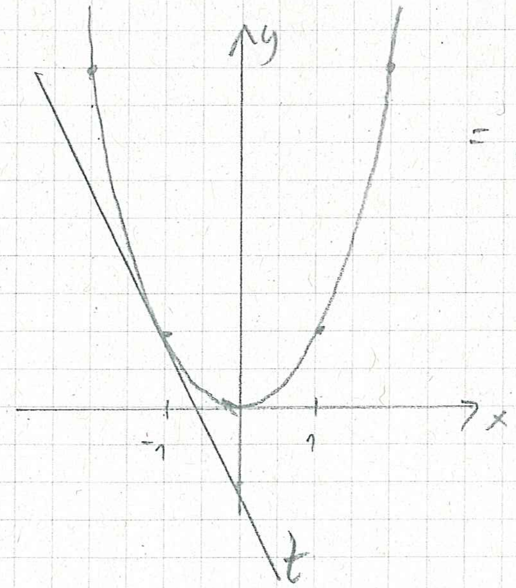


ex 3



pente de la tangente à f en $(-1;1)$

$$= f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)}$$

$$= \lim_{x\to -1} \frac{x^2-(-1)^2}{x+1}$$

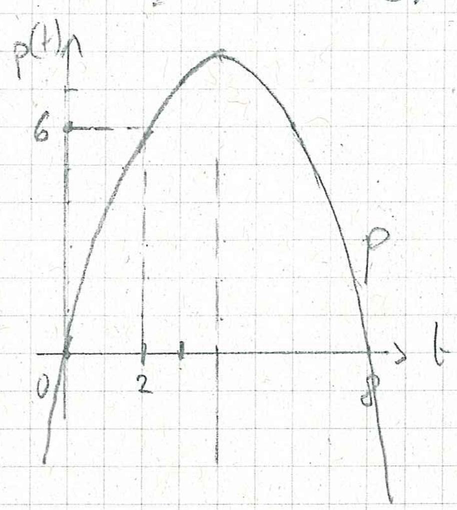
$$= \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$= \lim_{x\to -1}(x-1) = -1-1 = -2$$

ex 4

$$p(t) = 4t - \frac{t^2}{2} = 4t\left(1-\frac{t}{8}\right) \qquad Z_p = \{0;8\}$$

sommet : $(4;p(4)) = (4;8)$



a) $v = \frac{\Delta p}{\Delta t}$ donc : (i) $\frac{p(3)-p(2)}{3-2} = \frac{7{,}5-6}{1} = 1{,}5$

(ii) $$\frac{p(t)-p(2)}{t-2} = \frac{\left[4t-\frac{t^2}{2}\right]-6}{t-2} = \frac{-\frac{t^2}{2}+4t-6}{t-2}$$

$$= \frac{-\frac{1}{2}\left[t^2-8t+12\right]}{t-2} = \frac{-\frac{1}{2}(t-6)\cancel{(t-2)}}{\cancel{t-2}} \underset{t\neq 2}{=} -\frac{1}{2}(t-6)$$

(iii) $$\frac{p(2+h)-p(2)}{(2+h)-2} = \frac{\left[4(2+h)-\frac{(2+h)^2}{2}\right]-6}{h}$$

$$= \frac{8+4h-\left[\frac{4+4h+h^2}{2}\right]-6}{h} = \frac{\cancel{8}+4h-\cancel{2}-2h-\frac{h^2}{2}-\cancel{6}}{h}$$

$$= \frac{2h-\frac{h^2}{2}}{h} = \frac{\cancel{h}\left(2-\frac{h}{2}\right)}{\cancel{h}} \underset{h\neq 0}{=} 2-\frac{h}{2}$$

(iv) $\frac{p(t)-p(a)}{t-a} = \frac{\left[4t-\frac{t^2}{2}\right]-\left[4a-\frac{a^2}{2}\right]}{t-a} = \frac{4t-4a-\frac{1}{2}\left[t^2-a^2\right]}{t-a}$

$= \frac{4(t-a)-\frac{1}{2}(t-a)(t+a)}{t-a} = \frac{(t-a)\left[4-\frac{1}{2}(t+a)\right]}{t-a}$

$\underset{t\neq a}{=} 4-\frac{1}{2}(t+a)$

(v) $\frac{p(t+h)-p(t)}{(t+h)-t} = \frac{\left[4(t+h)-\frac{(t+h)^2}{2}\right]-\left[4t-\frac{t^2}{2}\right]}{t+h-t}$

$= \frac{\left[4t+4h-\frac{t^2+2th+h^2}{2}\right]-4t+\frac{t^2}{2}}{h}$

$= \frac{\cancel{4t}+4h-\cancel{\frac{t^2}{2}}-th-\frac{h^2}{2}-\cancel{4t}+\cancel{\frac{t^2}{2}}}{h} \underset{h\neq 0}{=} \frac{\cancel{h}(4-t-\frac{h}{2})}{\cancel{h}} = 4-t-\frac{h}{2}$

b) V instantanée $= \frac{dp}{dt}$ où d signifie "accroissement infinitésimal"

(i) Vinst en $t=2$:

en partant de a)(ii) : $p'(2)=\lim_{t\to 2}\frac{p(t)-p(2)}{t-2}$

$=\lim_{t\to 2} -\frac{1}{2}(t-6) = -\frac{1}{2}(-4) = 2$

en partant de a)(iii) : $p'(2) = \lim_{h\to 0}\frac{p(2+h)-p(2)}{h}$

$=\lim_{h\to 0}\left(2-\frac{h}{2}\right) = 2-\frac{0}{2} = 2$

(ii) Vinst en t :

en partant de a)(iv) : $p'(a) = \lim_{t\to a}\frac{p(t)-p(a)}{t-a}$

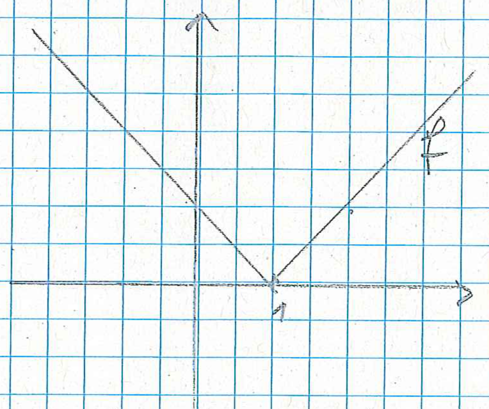
$= \lim_{t\to a} 4-\frac{1}{2}(t+a) = 4-\frac{1}{2}(a+a) = 4-a$

d'où $p'(a) = 4-a$ qu'on renomme en $p'(t) = 4-t$

en partant de a)(v) : $p'(t) = \lim_{h\to 0}\frac{p(t+h)-p(t)}{h}$

$= \lim_{h\to 0}\left(4-t-\frac{h}{2}\right) = 4-t$ direct !

ex5 a)



$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{|x-1|-|1-1|}{x-1} = \lim_{x\to 1} \frac{|x-1|}{x-1}$

on doit connaître : $|x-1| = \begin{cases} x-1 & \text{si } x \geq 1 \\ -(x-1) & \text{si } x < 1 \end{cases}$

et donc traiter $x \to 1^-$ et $x \to 1^+$:

$\lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{\cancel{x-1}}{\cancel{x-1}} = \lim_{x\to 1^+} \frac{1}{1} = 1$

$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \lim_{x\to 1^-} \frac{-(\cancel{x-1})}{\cancel{x-1}} = \lim_{x\to 1^-} \frac{-1}{1} = -1$

donc $\lim_{x\to 1} \frac{|x-1|}{x-1} \nexists$ càd $f'(1) \nexists$

Int. géom : la sécante passant par $(1;0)$ et un autre point proche de la courbe de f a une pente qui tend vers $\begin{cases} 1 & \text{quand } x \to 1^+ \\ -1 & \text{''} \quad x \to 1^- \end{cases}$

Par contre :

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x-1|-|0-1|}{x}$

$= \lim_{x\to 0} \frac{|x-1|-1}{x} = \lim_{x\to 0} \frac{-(x-1)-1}{x} = \lim_{x\to 0} \frac{-x+1-1}{x}$

car $|x-1| = -(x-1)$ quand $x \to 0$

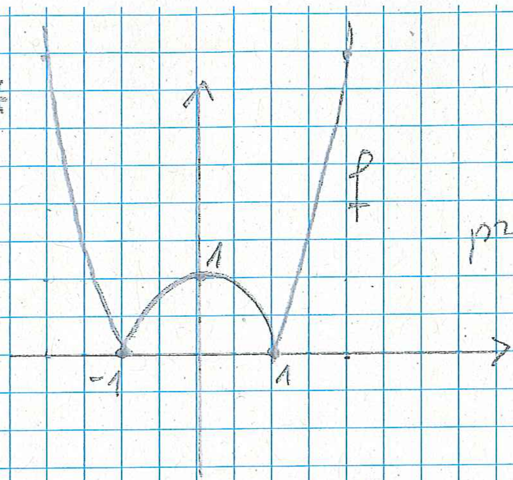
$= \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} \frac{-1}{1} = -1$ [la pente de la tg à f en $(0;1)$ vaut -1]

$f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{|x-1|-|2-1|}{x-2} = \lim_{x\to 2} \frac{|x-1|-1}{x-2}$

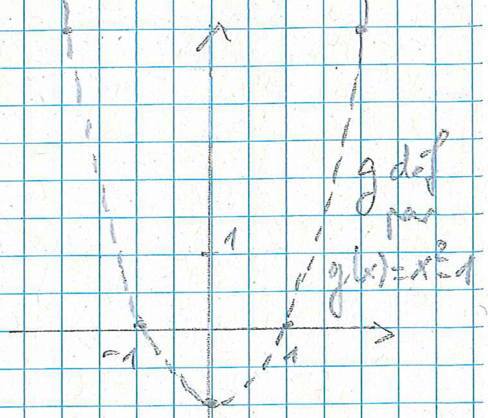
$= \lim_{x\to 2} \frac{(x-1)-1}{x-2} = \lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2} \frac{1}{1} = 1$

car $|x-1| = (x-1)$ qd $x \to 2$

[la pente de la tg à f en $(2;1)$ vaut 1] ex5   b)*



provient de :



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{|(x+h)^2-1|-|x^2-1|}{h}$$

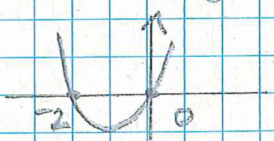
$$=\lim_{h\to 0}\frac{|x^2+2xh+h^2-1|-|x^2-1|}{h}\ \ldots$$

pour simplifier, étudions en $x=1$:

$$f'(1)=\lim_{h\to 0}\frac{|1+2h+h^2-1|-|0|}{h}=\lim_{h\to 0}\frac{|h^2+2h|}{h}$$

pour contrôler $|h^2+2h|$, il faut connaître son signe :

$g(h)=h^2+2h=h(h+2)$



est ≥ 0 si $h\in\mathbb{R}\setminus]-2;0[$

<0 si $h\in]-2;0[$

donc $|h^2+2h|=\begin{cases} h^2+2h & \text{si } h\notin]-2;0[\\ -(h^2+2h) & \text{si } h\in]-2;0[\end{cases}$

et on doit considérer $h\to 0^+$ et $h\to 0^-$

$$\lim_{h\to 0^+}\frac{|h^2+2h|}{h}=\lim_{h\to 0^+}\frac{h^2+2h}{h}=\lim_{h\to 0^+}\frac{\cancel{h}(h+2)}{\cancel{h}}=2$$

$$\lim_{h\to 0^-}\frac{|h^2+2h|}{h}=\lim_{h\to 0^-}\frac{-(h^2+2h)}{h}=\lim_{h\to 0^-}\frac{-\cancel{h}(h+2)}{\cancel{h}}=-2$$

donc $\lim_{h\to 0}\frac{|h^2+2h|}{h}=f'(1)\ \nexists$

Int^t géom : qd $h\to 0^+$, la pente de la sécante tend vers 2

" $h\to 0^-$, " " " " " " " " -2

ex6

$$f(t) = t^{10} - 7t^2 + 3$$

$$f'(-1) = \lim_{t \to -1} \frac{f(t) - f(-1)}{t - (-1)} = \lim_{t \to -1} \frac{(t^{10} - 7t^2 + 3) - (1 - 7 + 3)}{t + 1}$$

$$= \lim_{t \to -1} \frac{t^{10} - 7t^2 + 6}{t + 1}$$

on sait que $t^{10} - 7t^2 + 6$ doit être divisible par $t+1$:

$$\begin{array}{l|l}
t^{10} - 7t^2 + 6 & t+1 \\
\hline
-\ (t^{10} + t^9) & t^9 - t^8 + t^7 - t^6 + t^5 - t^4 + t^3 - t^2 - 6t + 6 \\
-t^9 - 7t^2 + 6 & \\
-\ (-t^9 - t^8) & \\
t^8 - 7t^2 + 6 & \\
-\ (t^8 + t^7) & \\
-t^7 - 7t^2 + 6 & \\
-\ (-t^7 - t^6) & \\
t^6 - 7t^2 + 6 & \\
-\ (t^6 + t^5) & \\
-t^5 - 7t^2 + 6 & \\
-\ (-t^5 - t^4) & \\
t^4 - 7t^2 + 6 & \\
-\ (t^4 + t^3) & \\
-t^3 - 7t^2 + 6 & \\
-\ (-t^3 - t^2) & \\
-6t^2 + 6 & \\
-\ (-6t^2 - 6t) & \\
6t + 6 & \\
-\ (6t + 6) & \\
0 &
\end{array}$$

d'où $f'(x) = \lim_{t \to -1} \frac{(t^9 - t^8 + t^7 - t^6 + t^5 - t^4 + t^3 - t^2 - 6t + 6)\cancel{(x+1)}}{\cancel{(x+1)}}$

$= (-1) - 1 + (-1) - 1 + (-1) - 1 + (-1) - 1 - 6(-1) + 6$

$= -8 + 6 + 6 = 4$