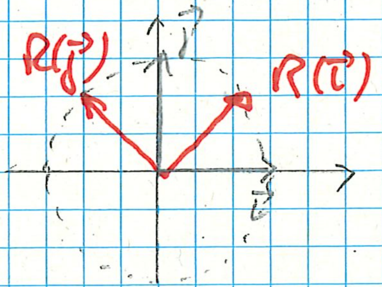
ex 39

a)



$R(\vec{i}) = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ $\quad M_R = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

$R(\vec{j}) = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$

b) $M_R^2 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

càd $M_R^2(\vec{i}) = \vec{j}$ et $M_R^2(\vec{j}) = -\vec{i}$

c'est la rotation d'angle $\pi/2$

(logique : rotation de $\frac{\pi}{4}$ puis de $\frac{\pi}{4}$ = rotation de $\frac{\pi}{2}$!)

Idem : $M_R^3 = \dots = \begin{pmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix} = M_{R_{\frac{3\pi}{4}}}$

$M_R^4 = \dots = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = M_{R_\pi}$

c) $M_R^{2016} = \left[(M_R)^4\right]^{504} = \left[M_{R_\pi}\right]^{504} = \left[(M_{R_\pi})^2\right]^{252}$

$= \left[M_{R_{2\pi}}\right]^{252} = Id^{252} = Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ex 38

(a) $L \circ K \begin{pmatrix} x \\ y \end{pmatrix} = L\left(K\begin{pmatrix} x \\ y \end{pmatrix}\right) = L\begin{pmatrix} 3x+y \\ x-2y \end{pmatrix}$

$= \begin{pmatrix} 2(3x+y) - (x-2y) \\ 3(3x+y) + 7(x-2y) \end{pmatrix} = \begin{pmatrix} 5x+4y \\ 16x-11y \end{pmatrix}$

(b) $\left.\begin{array}{l} L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix} \end{array}\right\}$ donc $M_L = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$

$\left.\begin{array}{l} K\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{array}\right\}$ donc $M_K = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$

$L \circ K \begin{pmatrix} x \\ y \end{pmatrix} = M_L \cdot M_K \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$= \begin{pmatrix} 6-1 & 2+2 \\ 9+7 & 3-14 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 16 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x+4y \\ 16x-11y \end{pmatrix}$

ex 42 a) Symétrie axiale d'axe Ox : $M_{S_{Ox}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\det(M_{S_{Ox}}) = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$

donc S_{Ox} inversible (car S_{Ox} bijective)

Symétrie axiale d'axe Oy : $M_{S_{Oy}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(M_{S_{Oy}}) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$

donc S_{Oy} inversible (car S_{Oy} bijective)

c)* Cas général : $S_\alpha = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$ matrice générale d'une symétrie d'axe passant par (0;0) et faisant un angle α avec Ox

$$\begin{vmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{vmatrix} = -\cos^2(2\alpha) - \sin^2(2\alpha)$$
$$= -[\underbrace{\sin^2(2\alpha) + \cos^2(2\alpha)}_{=1}] = -1 \neq 0$$

donc S_α bij inversible

Rem : car S_α bij bijective...

b) Projection sur l'axe Ox : $M_{P_{Ox}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $|M_{P_{Ox}}| = 1\cdot 0 - 0\cdot 0 = 0$

" " " Oy : $M_{P_{Oy}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $|M_{P_{Oy}}| = 0\cdot 1 - 0\cdot 0 = 0$

donc non inversibles, car non bijectives...

ex 41 a) $A = \begin{pmatrix} 4 & -1 \\ -2 & 5 \end{pmatrix}$ $A^{-1} = \frac{1}{22}\begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$ $A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5x+y}{22} \\ \frac{2x+4y}{22} \end{pmatrix}$

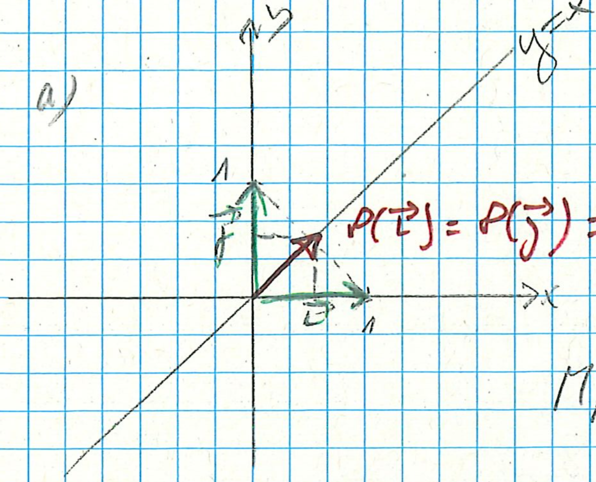
b) $B = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$ $B^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ $B^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3x+y}{3} \\ y/3 \end{pmatrix}$

c) $C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $C^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$ $C^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 x + 1/2 y \\ 1/2 x - 1/2 y \end{pmatrix}$

d) $D = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$ $\det D = 0$ $D^{-1} \nexists$

ex 43

a)



$P(\vec{i}) = P(\vec{j}) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$

$M_p = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$

b) $M_p^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = M_p$

(la 2e projection ne change plus rien !)

c) $M_p^n = M_p$ (idem pour la n-ième)

d) $\det M_p = |M_p| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = 0$ donc $M_p^{-1} \nexists$

car M_p n'est pas bijective !

ex44 S_1 : sym d'axe $y=x \Rightarrow M_{S_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

S_2 : " " $y=-x \Rightarrow M_{S_2} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$S_2 \circ S_1 = M_{S_2} \cdot M_{S_1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

càd $S_2 \circ S_1(\vec{i}) = -\vec{i}$ et $S_2 \circ S_1(\vec{j}) = -\vec{j}$

et, en général : $S_2 \circ S_1 \begin{pmatrix} x \\ y \end{pmatrix} = S_2 \circ S_1 (x\vec{i} + y\vec{j})$

$= x \, S_2 \circ S_1(\vec{i}) + y \cdot S_2 \circ S_1(\vec{j})$

$= x \cdot (-\vec{i}) + y(-\vec{j}) = \begin{pmatrix} -x \\ -y \end{pmatrix} = R_{180} \begin{pmatrix} x \\ y \end{pmatrix}$

ex45 $M_{S_x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_{S_y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $M_{R_{90}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $M_{R_{180}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

a) $S_x \circ S_y = M_{S_x} \cdot M_{S_y} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = R_{180}$

b) $S_y \circ R_{180} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = S_x$

c) $R_{90} \circ S_x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: symétrie d'axe $y=x$

d) $R_{180} \circ R_{90} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: rotation $270°$

ex46 $S_{30} = \begin{pmatrix} \cos(60) & \sin(60) \\ \sin(60) & -\cos(60) \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

$S_{60} = \begin{pmatrix} \cos(120) & \sin(120) \\ \sin(120) & -\cos(120) \end{pmatrix} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

$S_{60} \circ S_{30} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} +1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} = \begin{pmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{pmatrix} = R_{60}$

ex47 $R_{45} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$ et $H = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

$H \circ R_{45} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

ex 48 $M_L = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} \Rightarrow M_L^{-1} = \dfrac{1}{13}\begin{pmatrix} 5 & -3 \\ 1 & 2 \end{pmatrix}$

$$\Rightarrow L^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & -3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x/13 - 3/13\, y \\ x/13 + 2/13\, y \end{pmatrix}$$

ex 49 $M_L = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ $\det M_L = 0 \Rightarrow L^{-1} \nexists$

ex 50 $M_{S_\alpha} = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} \Rightarrow \det(M_{S_\alpha}) = -\cos^2(2\alpha) - \sin^2(2\alpha) = -1$

$$M_{S_\alpha}^{-1} = -1\begin{pmatrix} -\cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{pmatrix} = \begin{pmatrix} \cos(2\alpha) & +\sin(2\alpha) \\ +\sin(2\alpha) & -\cos(2\alpha) \end{pmatrix} = M_{S_\alpha}$$

ex 51 $M_p = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \Rightarrow \det M_p = 0 \Rightarrow M_p^{-1} \nexists$

ex 52 $A = M_{R_{120}} \Rightarrow M_{R_{120}} \circ M_{R_{120}} \circ M_{R_{120}} = Id$

càd $A^3 = A$

et on a ; $M_{R_{120}} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$

ex 53 $S_1 = \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}$ et $S_2 = \begin{pmatrix} \cos(2\beta) & \sin(2\beta) \\ \sin(2\beta) & -\cos(2\beta) \end{pmatrix}$

$$S_1 \circ S_2 = \begin{pmatrix} \cos(2\alpha)\cos(2\beta) + \sin(2\alpha)\sin(2\beta) & \cos(2\beta)\sin(2\beta) - \sin(2\alpha)\cos(2\beta) \\ \sin(2\alpha)\cos(2\beta) - \cos(2\alpha)\sin(2\beta) & \sin(2\alpha)\sin(2\beta) + \cos(2\alpha)\cos(2\beta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\alpha - 2\beta) & -\sin(2\alpha - 2\beta) \\ \sin(2\alpha - 2\beta) & \cos(2\alpha - 2\beta) \end{pmatrix}$$

$$= R_{2\alpha - 2\beta}$$