Act 17 $E(X)$ et $V(X)$ pour $X \sim N(0;1)$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \left(-e^{-x^2/2}\right)\Big|_{-\infty}^{\infty} = 0$$

$$V(X) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - [E(X)]^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx - 0^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot \left(x e^{-x^2/2}\right) dx = \frac{1}{\sqrt{2\pi}} \left[-x e^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \right]$$

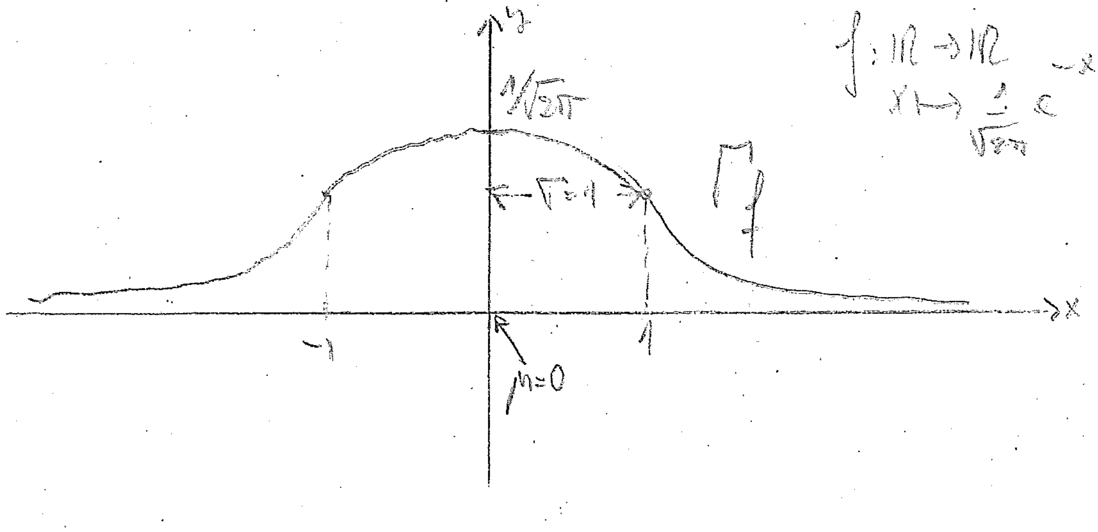
par parties :
$g(x) = x \Rightarrow g'(x) = 1$
$h'(x) = x e^{-x^2/2} \Rightarrow h(x) = -e^{-x^2/2}$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \left(-\frac{x}{e^{x^2/2}} \Big|_{-\infty}^{\infty} \right)}_{=0} + \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx}_{=1} = 1$$

("exp gagne sur pol" ou thm l'Hospital)

on l'a admis précédemment

donc $\sigma(X) = \sqrt{V(X)} = 1$



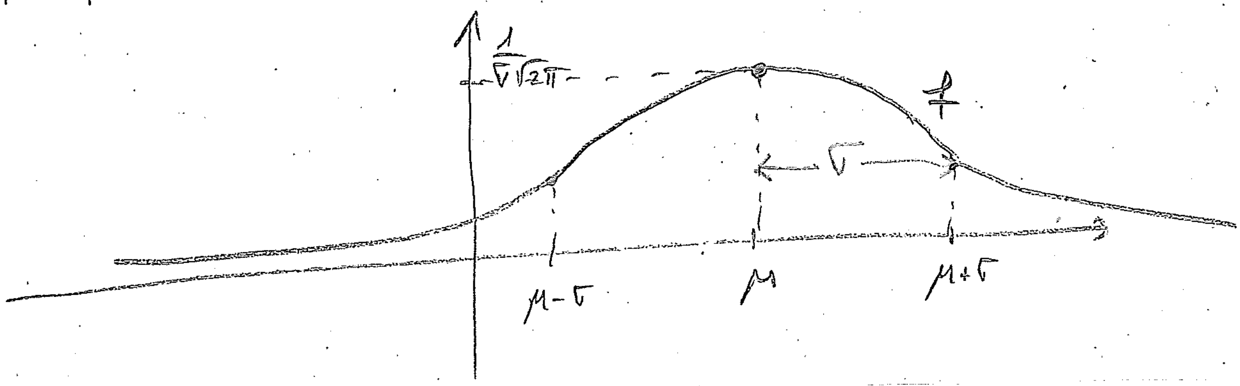
$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Ex 18

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}$$

Même principe que pour $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$:

- $D_f = \mathbb{R}$; zeros $= \emptyset$; $f(0) = \frac{1}{\sigma\sqrt{2\pi}}$
- as. horiz $y=0$ à $\pm\infty$
- max à $(\mu ; f(\mu))$
- pts infl. à $(\mu \pm \sigma ; f(\mu \pm \sigma))$



f est une fonction de densité ?

A voir : [1] $\int_{-\infty}^{\infty} f(z)\,dz \overset{?}{=} 1 \Leftrightarrow \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-\mu}{\sigma}\right)^2/2} dz \overset{?}{=} 1$

chg. de var : $x = \frac{z-\mu}{\sigma}$: $dx = \frac{1}{\sigma} dz \Leftrightarrow dz = \sigma dx$

si $x \to +\infty$: $z \to +\infty$ (car $\sigma > 0$)
si $x \to -\infty$: $z \to -\infty$

$\Leftrightarrow \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \cancel{\sigma} dx \overset{?}{=} 1$ ok (cf $N(0;1)$)

[2] $f(z) \geq 0$ ✓

Act 19

$\boxed{1}$ $Z \sim N(2; 1{,}4)$

(a) i) $P(Z \leq 1{,}43) \underset{c}{\cong} 0{,}3420$

ii) $P(Z \leq -0{,}97) \underset{c}{\cong} 0{,}0163$

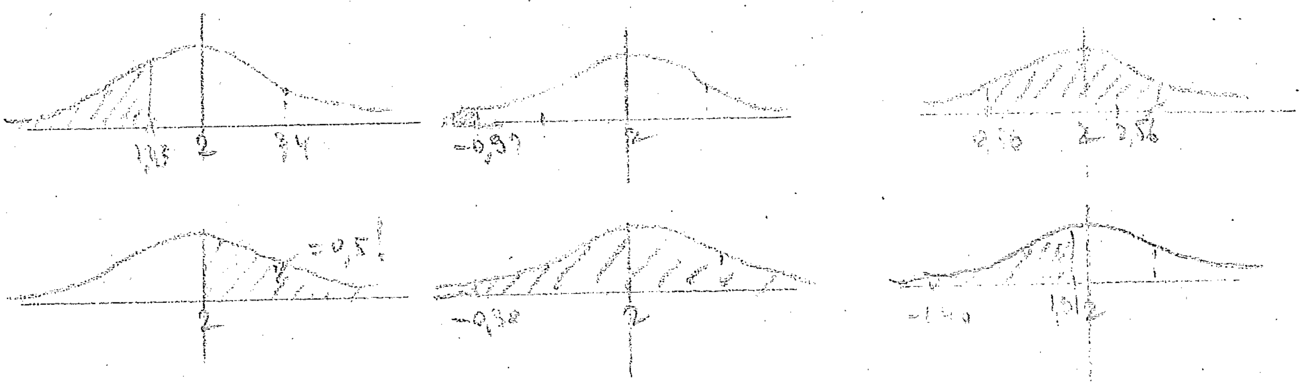
iii) $P(0{,}56 \leq Z \leq 2{,}56) \underset{c}{\cong} 0{,}5036$

iv) $P(2 \leq Z) = 1 - P(Z < 2)$

$= 1 - 0{,}5 = 0{,}5$

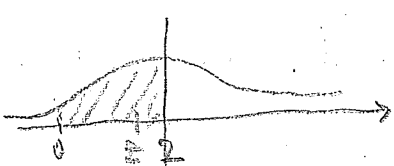
v) $P(Z \geq -0{,}38) = 1 - P(Z < -0{,}38) \underset{c}{\cong} 0{,}8554$

vi) $P(-1{,}46 \leq Z \leq 1{,}91) \underset{c}{\cong} 0{,}4696$



(b) $P(Z \leq 1{,}43) = P\left(\frac{Z-2}{1{,}4} \leq \frac{1{,}43-2}{1{,}4}\right) = P\Big(\underbrace{\frac{Z-2}{1{,}4}}_{\sim N(0,1)\,!} \leq -0{,}4071\Big) \cong 0{,}3420$

$\boxed{4}$ $P(0 \leq Z \leq z) = 0{,}4390$



on table : $P(0 \leq Z \leq 2) \underset{c}{\cong} 0{,}4234 < 0{,}4390$

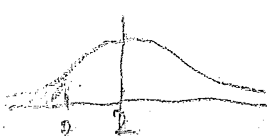
donc $z > 2$

d'où $P(0 \leq Z \leq z) = P(Z \leq z) - P(Z \leq 0) = 0{,}4390$

$\Leftrightarrow P(Z \leq z) = \underbrace{P(Z \leq 0)}_{0{,}07656} + 0{,}4390 \cong 0{,}5136$

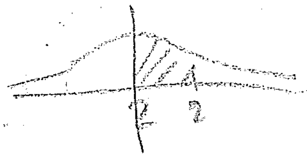
$\underset{c}{\Leftrightarrow} z \cong 2{,}0477$

$\boxed{3}$



$P(Z < z) = 0{,}1453 \underset{c}{\Leftrightarrow} z \cong 0{,}5205$

$\boxed{2}$



$P(2 < Z < z) = 0{,}0432 \Leftrightarrow P(Z < z) - \underbrace{P(Z \leq 2)}_{= 0{,}5} = 0{,}0432$

$\Leftrightarrow P(Z < z) = 0{,}4568 \underset{c}{\Leftrightarrow} z \cong 1{,}8481$

Act 21 old

$\boxed{1}$ $X \sim N(20;3)$ $P(21 \leq X \leq 26) = P\left(\frac{21-20}{3} \leq Z \leq \frac{26-20}{3}\right)$

$= P\left(\frac{1}{3} \leq Z \leq 2\right) = P(Z \leq 2) - P\left(Z \leq \frac{1}{3}\right)$

calc: $p \simeq 0,346791...$

$\simeq 0,9772 - 0,63 = 0,3472 \simeq 34,7\ \%$

$\boxed{2}$ $X \sim N(173;8)$ $P(160 \leq X \leq 175) = P\left(\frac{160-173}{8} \leq Z \leq \frac{175-173}{8}\right)$

$= P(-1,625 \leq Z \leq 0,25) = P(Z \leq 0,25) - P(Z \leq -1,625)$

$p \simeq 0,546625...$

$= P(X \leq 0,25) - [1 - P(X \leq 1,625)]$

$\simeq 0,5987 - 1 + 0,9479 \simeq 0,5466 \simeq 54,7\ \%$

TM 052

Act 20 $E(X)$ et $V(X)$ pour $X \sim N(\mu;\sigma)$

$$E(X) \overset{\text{déf}}{=} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

on pose $y = \frac{x-\mu}{\sigma} \Leftrightarrow x = \sigma y + \mu$; $x \to \pm\infty \Rightarrow y \to \pm\infty$ $(\sigma > 0)$

$dx = \sigma\, dy$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu)\, e^{-1/2 y^2} \cancel{\sigma}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma y\, e^{-y^2/2} dy + \frac{1}{\sqrt{2\pi}} \mu \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$= \sigma \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y\, e^{-y^2/2} dy\right]}_{= 0 \text{ (espérance pour } N(0;1))} + \mu \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy\right]}_{= 1 \text{ (fct de densité pour } N(0;1))} = \mu$$

$$V(X) \overset{\text{thm}}{=} E(X^2) - \mu^2 = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx - \mu^2$$

même subst. $= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma y + \mu)^2 e^{-y^2/2} \cancel{\sigma}\, dy - \mu^2$

$$= \sigma^2 \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy}_{= 1 \text{ (variance pour } N(0;1))} + 2\sigma\mu \underbrace{\frac{1}{\sqrt{2\pi}} \int y\, e^{-y^2/2} dy}_{= 0 \text{ (esp pour } N(0;1))}$$

$$+ \mu^2 \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy}_{= 1 \text{ (fct densité pour } N(0;1))} - \mu^2$$

$$= \sigma^2$$

Aul 21

$\boxed{1}$ $X \sim N(20,3)$

$P(21 \leq X \leq 26) \mathrel{\widehat{=}} 0{,}3467 \approx 34{,}7\,\%$

$\boxed{2}$ $X \sim N(173,8)$

$P(160 \leq X \leq 175) \mathrel{\widehat{=}} 0{,}5466 \approx 54{,}7\,\%$