ex10 $\quad M_{ee}(L) = \begin{pmatrix} 2 & 8 \\ -3 & -12 \end{pmatrix}$ c-à-d $\begin{cases} L(\vec{i}) = 2\vec{i} - 3\vec{j} \\ L(\vec{j}) = 8\vec{i} - 12\vec{j} \end{cases}$

(i) Ker(L): $M_{ee}(L)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2x+8y=0 \\ -3x-12y=0 \end{cases} \Leftrightarrow \begin{cases} x=-4y \\ x=-4y \end{cases}$

$$Ker(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x=-4y \right\} = \left\{ \begin{pmatrix} -4y \\ y \end{pmatrix} \middle| y \in \mathbb{R} \right\} = \left\{ y\begin{pmatrix} -4 \\ 1 \end{pmatrix} \middle| y \in \mathbb{R} \right\}$$

Im(L): $M_{ee}(L)\cdot\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{cases} 2x+8y = x' \\ -3x-12y = y' \end{cases} \begin{vmatrix} 3 \\ 2 \end{vmatrix}$

$$0 = 3x' + 2y'$$
$$y' = -\frac{3}{2}x'$$

$$Im(L) = \left\{ \begin{pmatrix} x' \\ y' \end{pmatrix} \middle| y' = -\tfrac{3}{2}x' \right\} = \left\{ \begin{pmatrix} x' \\ -\frac{3}{2}x' \end{pmatrix} \middle| x \in \mathbb{R} \right\} = \left\{ x'\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$$

plus simple $= \left\{ x'\begin{pmatrix} 2 \\ -3 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$

(ii) $P_L(\lambda) = \begin{vmatrix} \lambda-2 & -8 \\ 3 & \lambda+12 \end{vmatrix} = (\lambda-2)(\lambda+12)+24 = \lambda^2+10\lambda = \lambda(\lambda+10)$

$\lambda_1 = -10$ val propre: $\begin{pmatrix} 2 & 8 \\ -3 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -10\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} 2x+8y=-10x \\ -3x-12y=-10y \end{cases}$

$$\Leftrightarrow \begin{cases} 12x+8y=0 \\ -3x-2y=0 \end{cases}$$

ss esp. propre: $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| 3x+2y=0 \right\} = \left\{ \begin{pmatrix} x \\ -\frac{3}{2}x \end{pmatrix} \middle| x \in \mathbb{R} \right\}$
$= \left\{ x\begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \middle| x \in \mathbb{R} \right\} = \left\{ x\begin{pmatrix} 2 \\ -3 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$

vect propre $\vec{v_1}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

$\lambda_2 = 0$ val propre: $\begin{pmatrix} 2 & 8 \\ -3 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} 2x+8y=0 \\ -3x-12y=0 \end{cases}$ (cf Ker(L))

ss esp. propre: $\left\{ y\begin{pmatrix} -4 \\ 1 \end{pmatrix} \middle| y \in \mathbb{R} \right\}$

vect propre $\vec{v_2}\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

Base propre: $B = (\vec{v_1}; \vec{v_2}) = \left(\begin{pmatrix} 2 \\ -3 \end{pmatrix}; \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right)$

(ou $B' = (\vec{v_2}; \vec{v_1}) = \left(\begin{pmatrix} -4 \\ 1 \end{pmatrix}; \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right)$)

<u>ex11</u> a) $\begin{pmatrix}1\\0\end{pmatrix} = \alpha\begin{pmatrix}1\\-1\end{pmatrix} + \beta\begin{pmatrix}1\\1\end{pmatrix} \Leftrightarrow \begin{cases} 1 = \alpha + \beta \\ 0 = -\alpha + \beta \end{cases}$

$1 = 2\beta$

$\beta = \frac{1}{2} \Rightarrow \alpha = \frac{1}{2}$

$\begin{pmatrix}0\\1\end{pmatrix} = \delta\begin{pmatrix}1\\-1\end{pmatrix} + \gamma\begin{pmatrix}1\\1\end{pmatrix} \Leftrightarrow \begin{cases} 0 = \delta + \gamma \\ 1 = -\delta + \gamma \end{cases}$

$1 = 2\gamma$

$\gamma = \frac{1}{2} \Rightarrow \delta = -\frac{1}{2}$

$\Rightarrow L\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}L\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}L\begin{pmatrix}1\\1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}-3\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$L\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}-3\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} \Rightarrow M_{CC}(L) = \begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}$

b) $p_L(\lambda) = \begin{vmatrix}\lambda & -3\\-1 & \lambda-2\end{vmatrix} = \lambda(\lambda-2) - 3 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

<u>$\lambda_1 = -1$</u>: $\begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = (-1)\begin{pmatrix}x\\y\end{pmatrix} \Leftrightarrow \begin{cases} 3y = -x \\ x + 2y = -y \end{cases} \Leftrightarrow \begin{cases} 3y = -x \\ x = -3y \end{cases}$

ss esp propre $\left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| x = -3y\right\} = \left\{\begin{pmatrix}-3y\\y\end{pmatrix} \middle| y \in \mathbb{R}\right\} = \left\{y\begin{pmatrix}-3\\1\end{pmatrix} \middle| y \in \mathbb{R}\right\}$

vect propre: $\vec{v_1}\begin{pmatrix}-3\\1\end{pmatrix}$

<u>$\lambda_2 = 3$</u>: $\begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 3\begin{pmatrix}x\\y\end{pmatrix} \Leftrightarrow \begin{cases} 3y = 3x \\ x + 2y = 3y \end{cases} \Leftrightarrow \begin{cases} x = y \\ x = y \end{cases}$

ss esp propre $\left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| x = y\right\} = \left\{\begin{pmatrix}x\\x\end{pmatrix} \middle| x \in \mathbb{R}\right\} = \left\{x\begin{pmatrix}1\\1\end{pmatrix} \middle| x \in \mathbb{R}\right\}$

vect propre: $\vec{v_2}\begin{pmatrix}1\\1\end{pmatrix}$

base propre: $B = \left(\begin{pmatrix}-3\\1\end{pmatrix}; \begin{pmatrix}1\\1\end{pmatrix}\right)$

$M_{BB}(L)$: $L\begin{pmatrix}-3\\1\end{pmatrix} = \begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}-3\\1\end{pmatrix} = \begin{pmatrix}3\\-1\end{pmatrix} = (-1)\cdot\begin{pmatrix}-3\\1\end{pmatrix} + 0\begin{pmatrix}1\\1\end{pmatrix}$

$L\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0 & 3\\1 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = 0\cdot\begin{pmatrix}-3\\1\end{pmatrix} + 3\begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow M_{BB}(L) = \begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}$

ex 12

a) $M_L = \begin{pmatrix}1&1\\2&2\end{pmatrix}$ et $M_F = \begin{pmatrix}-3&0\\0&-3\end{pmatrix}$

$L\circ L = M_L \cdot M_L = \begin{pmatrix}1&1\\2&2\end{pmatrix}\begin{pmatrix}1&1\\2&2\end{pmatrix} = \begin{pmatrix}3&3\\6&6\end{pmatrix}$

$F\circ L = M_F \cdot M_L = \begin{pmatrix}-3&0\\0&-3\end{pmatrix}\begin{pmatrix}1&1\\2&2\end{pmatrix} = \begin{pmatrix}-3&-3\\-6&-6\end{pmatrix}$

$Ker(L\circ L) : \begin{pmatrix}3&3\\6&6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{cases}3x+3y=0\\6x+6y=0\end{cases} \quad x=-y$

$Ker(L\circ L) = \left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| x=-y\right\} = \left\{\begin{pmatrix}-y\\y\end{pmatrix} \middle| y\in\mathbb{R}\right\} = \left\{y\begin{pmatrix}-1\\1\end{pmatrix} \middle| y\in\mathbb{R}\right\}$

$Ker(F\circ L) : \begin{pmatrix}-3&-3\\-6&-6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Leftrightarrow \begin{cases}-3x-3y=0\\-6x-6y=0\end{cases} \quad x=-y$

$Ker(F\circ L) = \left\{y\begin{pmatrix}-1\\1\end{pmatrix} \middle| y\in\mathbb{R}\right\}$

b) $\lambda_1 = 0$ vecteur propre de $L\circ L$ et de $F\circ L$ / $\vec{v_1}\begin{pmatrix}-1\\1\end{pmatrix}$ vect pr.

$p_L(\lambda) = \begin{vmatrix}\lambda-3 & -3\\ -6 & \lambda-6\end{vmatrix} = (\lambda-3)(\lambda-6)-18 = \lambda^2-9\lambda = \lambda(\lambda-9)$

$\lambda_1 = 0$ et $\lambda_2 = 9$ val. pr.

$\lambda_2 = 9 : \begin{pmatrix}3&3\\6&6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 9\begin{pmatrix}x\\y\end{pmatrix}$

$\begin{cases}3x+3y=9x\\6x+6y=9y\end{cases} \Leftrightarrow \begin{cases}3y=6x\\6x=3y\end{cases} \quad y=2x$

ss esp propre : $\left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| y=2x\right\} = \left\{\begin{pmatrix}x\\2x\end{pmatrix} \middle| x\in\mathbb{R}\right\} = \left\{x\begin{pmatrix}1\\2\end{pmatrix} \middle| x\in\mathbb{R}\right\}$

$\vec{v_2}\begin{pmatrix}1\\2\end{pmatrix}$ vecteur propre

Base propre de $L\circ L = \left\{\begin{pmatrix}-1\\1\end{pmatrix}; \begin{pmatrix}1\\2\end{pmatrix}\right\}$; idem pour $F\circ L$

ex 13

$M = \begin{pmatrix}-2&6\\-2&5\end{pmatrix}$ cf théorie p37, on a : $B\left(\begin{pmatrix}2\\1\end{pmatrix};\begin{pmatrix}3\\2\end{pmatrix}\right)$ base propre

et $M_{BB}(L) = \begin{pmatrix}1&0\\0&2\end{pmatrix}$

$P_{CB} = \begin{pmatrix}2&3\\1&2\end{pmatrix}$, donc $P_{BC} = P_{CB}^{-1} = \frac{1}{1}\begin{pmatrix}2&-3\\-1&2\end{pmatrix} = \begin{pmatrix}2&-3\\-1&2\end{pmatrix}$

et $M = M_{CC}(L) = P_{CB} \cdot M_{BB}(L) \cdot P_{BC}$

donc $M^n = [P_{CB} \cdot M_{BB}(L) P_{BC}]^n$

$= (P_{CB}\, M_{BB}(L) \underbrace{P_{BC})(P_{CB}}_{Id} M_{BB}(L) \underbrace{P_{BC}) \ldots (P_{CB}}_{Id} M_{BB}(L) P_{BC})$

$= P_{CB} \cdot M_{BB}(L)^n \cdot P_{BC} = \begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}^n\begin{pmatrix}2&-3\\-1&2\end{pmatrix}$

$= \begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}1&0\\0&2^n\end{pmatrix}\begin{pmatrix}2&-3\\-1&2\end{pmatrix} = \begin{pmatrix}2&3\cdot2^n\\1&2\cdot2^n\end{pmatrix}\begin{pmatrix}2&-3\\-1&2\end{pmatrix} = \begin{pmatrix}4-3\cdot2^n & -6+6\cdot2^n\\ 2-2^{n+1} & -3+4\cdot2^n\end{pmatrix}$

ex 14 $\quad M_L = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix} = M_{cc}(L)$

on cherche N telle que $N^3 = M$

$P_L(\lambda) = (\lambda+5)(\lambda+2) - 18 = \lambda^2 + 7\lambda - 8 = (\lambda-1)(\lambda+8)$

donc $\lambda_1 = 1$ et $\lambda_2 = -8$ valeurs propres et B une base propre.

donc $M_{BB}(L) = \begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix}$ diagonalisation de $M_L = M_{cc}(L)$

Pour $M_{BB}(L)$, on voit facilement que $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ est telle que $A^3 = M_{BB}(L)$ [on est dans la base B]

puis $P_{BC}^{-1} A \cdot P_{BC} = D$ une certaine matrice qu'il n'est pas besoin de calculer !

et $\left[P_{BC}^{-1} A \, P_{BC}\right]^3 = D^3$

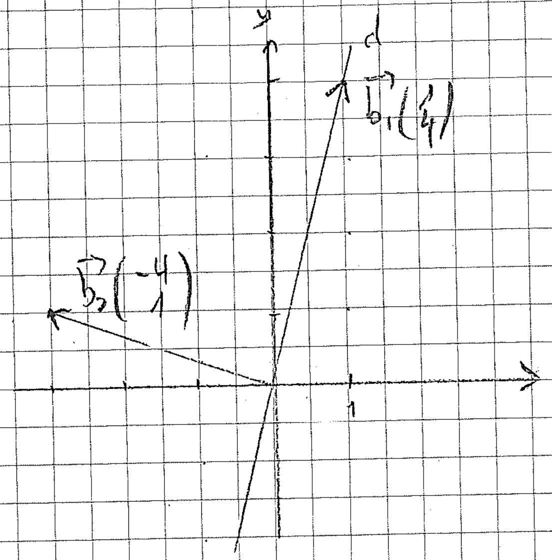
$\Leftrightarrow \quad P_{BC}^{-1} A \underbrace{P_{BC} P_{BC}^{-1}}_{I} A \underbrace{P_{BC} P_{BC}^{-1}}_{I} A P_{BC} = D^3$

$\Leftrightarrow \quad P_{BC}^{-1} \cdot A^3 \cdot P_{BC} = D^3$

$\Leftrightarrow \quad \underbrace{P_{BC}^{-1} \cdot M_{BB}(L) \cdot P_{BC}}_{M_{cc}(L) = M_L} = D^3$

càd $\quad M_L = D^3 \quad$ cqfd

ex 15



S_d : matrice compliquée à gérer dans la base canonique C ...

$M_R \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$

$L = R \circ S$

$M_L = M_R \, M_S$

(a) Pour mieux "voir" L, on veut travailler dans la base propre de S

on choisit $\vec{b}_1\begin{pmatrix}1\\4\end{pmatrix}$ et $\vec{b}_2\begin{pmatrix}-4\\1\end{pmatrix}$ qui sont vecteurs propres de S :

en effet, on a $S(\vec{b}_1) = 1 \cdot \vec{b}_1$ (val propre $\lambda_1 = 1$)

$S(\vec{b}_2) = -\vec{b}_2$ (val propre $\lambda_2 = -1$)

donc la base propre est $B(\vec{b}_1 ; \vec{b}_2)$

$\Rightarrow M_{BB}(S) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Aussi : $M_{CB} = P = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix}$ donc $M_{BC} = P^{-1} = \frac{1}{17}\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

$$\begin{tikzcd} B \arrow[r, "M_{BB}(S)"] \arrow[d, shift right, "P"'] & B \arrow[d, shift right, "P"'] \\ C \arrow[r, "S"] \arrow[u, shift right, "P^{-1}"'] \arrow[r, bend right, "M_{CC}(S)"'] & C \arrow[u, shift right, "P^{-1}"'] \end{tikzcd}$$

d'où $M_{CC}(S) = P \cdot M_{BB}(S) \cdot P^{-1} = \begin{pmatrix} 1 & -4 \\ 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{17}\begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 4 & -1 \end{pmatrix} \cdot \frac{1}{17}\begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} -15 & +8 \\ +8 & +15 \end{pmatrix} = \begin{pmatrix} -15/17 & +8/17 \\ +8/17 & +15/17 \end{pmatrix}$

(b) $M_{CC}(L) = M_{CC}(R) \cdot M_{CC}(S) = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \cdot \begin{pmatrix} -15/17 & 8/17 \\ 8/17 & 15/17 \end{pmatrix}$

$= \begin{pmatrix} -\frac{23\sqrt{2}}{34} & -\frac{7\sqrt{2}}{34} \\ -\frac{7\sqrt{2}}{34} & +\frac{23\sqrt{2}}{34} \end{pmatrix}$