Matu 2013 Corrigé

[/13]

Ex 1

(a) $\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1| + C$ (3)

(b) $\int \frac{x^2-1}{x}\,dx = \int \frac{x^2}{x}\,dx - \int \frac{1}{x}\,dx = \frac{x^2}{2} - \ln(x) + C$ (3)

(c) $\int x\cos(x)\,dx \underset{\uparrow}{=} x\sin(x) - \int \sin(x)\,dx$

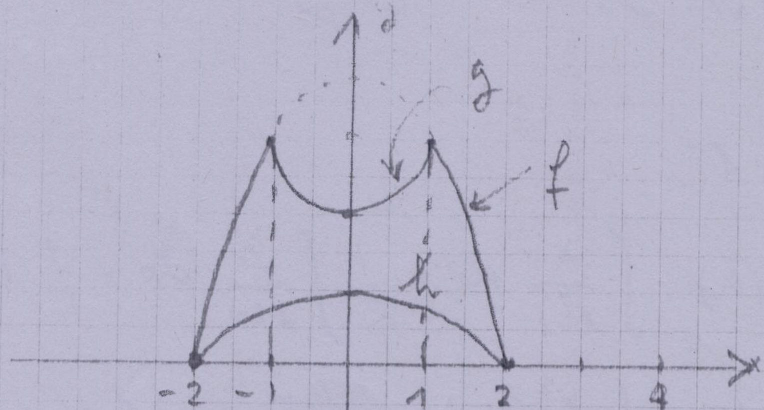
$f(x) = x \Rightarrow f'(x) = 1$
$g'(x) = \cos(x) \Rightarrow g(x) = \sin(x)$

$= x\sin(x) - (-\cos(x)) + C$
$= x\sin(x) + \cos(x) + C$ (4)

(d) $\int x\cos(x^2)\,dx = \frac{1}{2}\int 2x\cdot\cos(x^2)\,dx$
$= \frac{1}{2}\sin(x^2) + C$ (3)

[11]

Ex 2



$$A = \int_{-2}^{-1} f(x) - h(x)\,dx + \int_{-1}^{1} g(x) - h(x) + \int_{1}^{2} f(x) - h(x)$$ (4)

$$= \int_{-2}^{-1} (4-x^2) - \left(-\frac{x^2}{4}+1\right) dx + \int_{-1}^{1} (x^2+2) - \left(-\frac{x^2}{4}+1\right) dx + \int_{1}^{2} (4-x^2) - \left(-\frac{x^2}{4}+1\right)$$

$$= \int_{-2}^{-1} -\frac{3x^2}{4}+3\,dx + \int_{-1}^{1} \frac{5x^2}{4}+1\,dx + \int_{1}^{2} -\frac{3x^2}{4}+3\,dx$$

$$= -\frac{1}{4}x^3+3x\Big|_{-2}^{-1} + \frac{5x^3}{12}+x\Big|_{-1}^{1} + -\frac{1}{4}x^3+3x\Big|_{1}^{2}$$ (4)

$$= \left(\frac{1}{4}-3\right) - (2-6) + \left(\frac{5}{12}+1\right) - \left(-\frac{5}{12}-1\right) + (-2+6) - \left(-\frac{1}{4}+3\right)$$

$$= \frac{1}{4} - 3 + 4 + \frac{10}{12} + 2 + 4 + \frac{1}{4} - 3$$

$$= 4 + \frac{1}{2} + \frac{5}{6}$$

$$= \frac{24+3+5}{6}$$

$$= \frac{32}{6}$$

$$= \frac{16}{3}$$

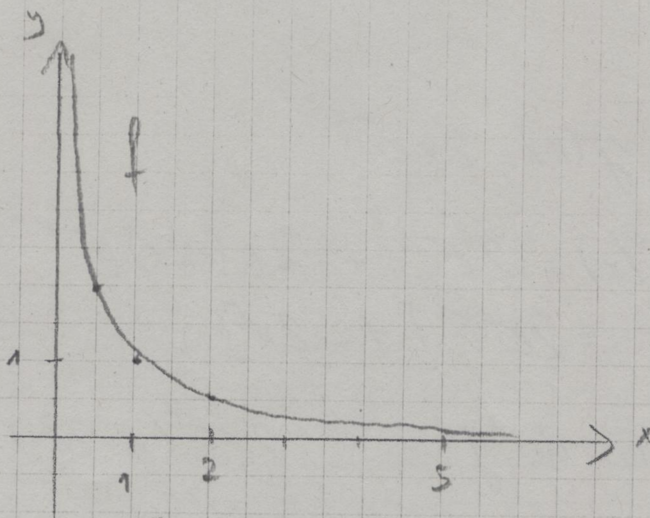
$$= 5,\bar{3}$$

$$\approx 5,3$$ (3)

Remarque : on aurait pu poser $A = 2\left[\int_0^1 g(x) - h(x) + \int_1^2 f(x) - h(x)\,dx\right]$

[1/12]

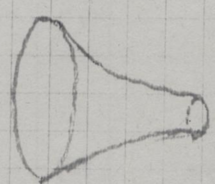
Ex 3



(a) $V = \pi \int_2^5 \left(\frac{1}{x}\right)^2 dx = \pi \left(-\frac{1}{x}\right)\Big|_2^5 = \pi\left[-\frac{1}{5} - \left(-\frac{1}{2}\right)\right]$

$= \pi\left[\frac{1}{2} - \frac{1}{5}\right] = \pi \frac{3}{10} = \frac{3\pi}{10} \cong 0{,}94$ (4)

(b)



(2)

(c) $V = \pi \int_a^b \pi \left(\frac{1}{x}\right)^2 dx = \pi \left(-\frac{1}{x}\right)\Big|_a^b = \pi\left(-\frac{1}{b} + \frac{1}{a}\right) = \pi \frac{b-a}{ab}$

(d) $\lim_{b \to +\infty} \pi \frac{b-1}{b} = \lim_{b \to +\infty} \pi \frac{\cancel{b}(1 - \overbrace{1/b}^{\to 0})}{\cancel{b}} = \pi \cdot 1 = \pi$ (2)

(e) $\lim_{a \to 0^+} \frac{\pi(5-a)}{5a} = \frac{5 \cdot \pi}{0^+} = +\infty$ (2)

[1/8]

Ex 4

$$f(x) = 2xe^{-x^2}$$

a) $f'(x) = 2e^{-x^2} + 2x(-2x)e^{-x^2}$

$= 2e^{-x^2}[1 - 2x^2]$ (3)

b) zeros de f'. $2e^{-x^2}(1-2x^2) = 0$

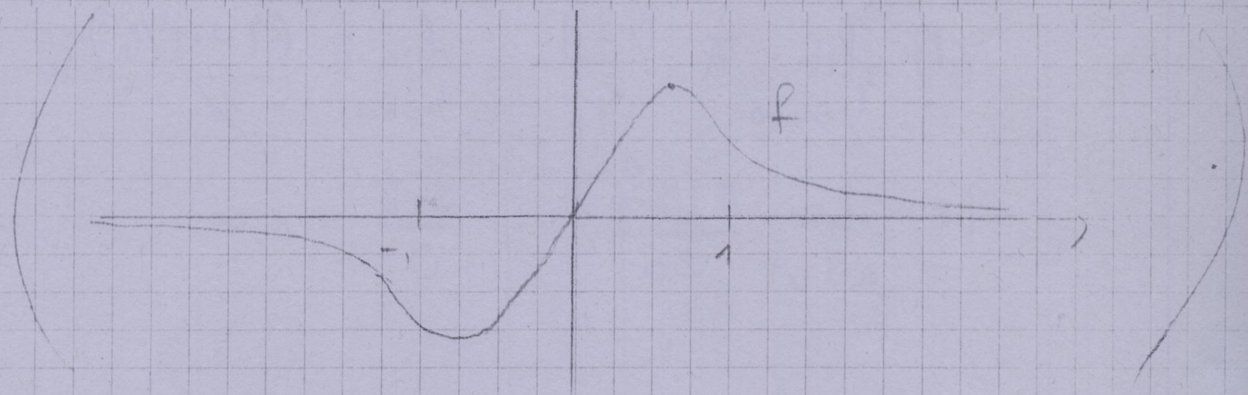
$e^{-x^2} = 0$ ou $1-2x^2 = 0$

$\emptyset$ $x = \pm\sqrt{1/2} \simeq \pm 0{,}71$ (2)

		$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$	
$2e^{-x^2}$	+	+	+	+	+
$1-2x^2$	−	0	+	0	−
$f'(x)$	−	0	+	0	−
$f(x)$	↘	m	↗	M	↘

(2)

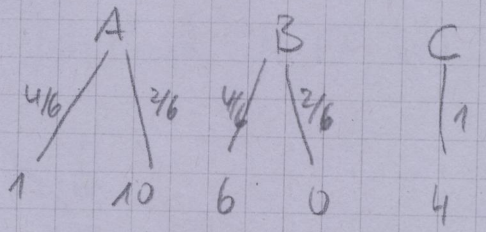
donc f est ↗ sur $\left[-\sqrt{\frac{1}{2}} ; \sqrt{\frac{1}{2}}\right]$ (1)



[/16]

Ex 5

on lance un dé



a) $P(\text{obt. } 10 \text{ avec dé } A) = \frac{2}{6} = \frac{1}{3} \simeq 33,3\ \%$ (1)

b) $B(6; \frac{4}{6}) \Rightarrow P(X=2) = C_2^6 \left(\frac{2}{6}\right)^2 \cdot \left(\frac{4}{6}\right)^4 \simeq 32,9\%$ (3)

c) $B(6; \frac{4}{6}) \Rightarrow P(X \geq 1) = 1 - P(X=0)$

$= 1 - \left(\frac{4}{6}\right)^6 \simeq 91,2\%$ (3)

d) $P(\text{"1 avec A" et "6 avec B"}) = \frac{4}{6} \cdot \frac{4}{6} = \frac{4}{9} \simeq 44,4\%$ (2)

e) A gagne si $A = 10$ $\quad p = 2/6$

$A = 1$ et $B = 0$ $\quad p = \frac{4}{6} \cdot \frac{2}{6} = \frac{2}{9}$

$p(A \text{ gagne}) = \frac{2}{6} + \frac{2}{9} = \frac{30}{54} = \frac{5}{9} \simeq 55,56\%$ (2)

f) B gagne si $B = 6$

$p(B \text{ gagne}) = \frac{4}{6} \simeq 66,67\ \%$ (1)

g) C gagne si $C = 4$ et $A : 1$ $\quad p = 1 \cdot \frac{4}{6} = \frac{4}{6}$

$p(C \text{ gagne}) = 1 \cdot \frac{4}{6} \simeq 66,67\ \%$ (1)

h) Non car si je choisis en 1er le dé A : C gagne contre A

B : A ---

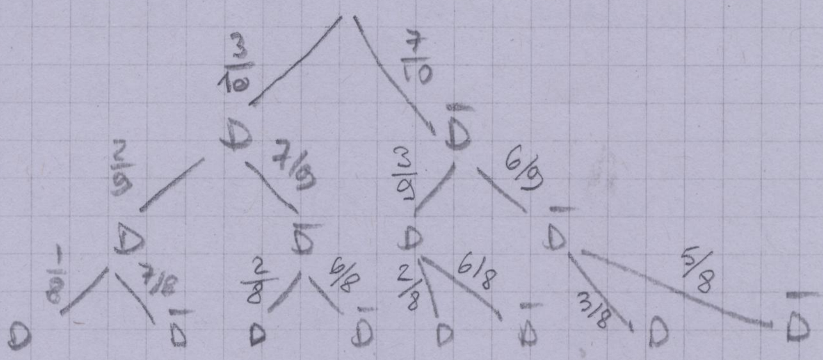
C : B --- (3)

Ex 6

[/14]

(a) $A^{10}_{3} = 10 \cdot 9 \cdot 8 = 720$ (3)

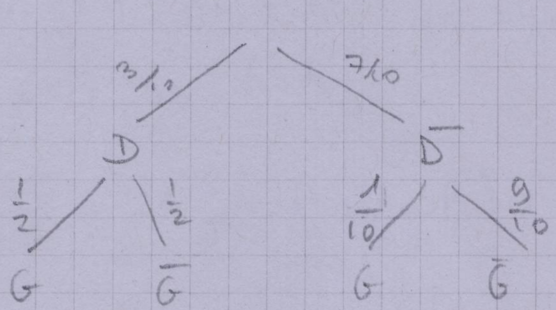
(b) P(1er choisit déf et 2e choisit déf et 3e choisit déf)



$P = \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{120} \simeq 0{,}83\,\%$ (3)

ou direct : $\frac{\text{choix de 3 machines defectueuses}}{\text{choix de 3 machines qcq}} : \frac{P_3}{A^{10}_3} = \frac{3!}{10 \cdot 9 \cdot 8}$

(c)



$P(D \mid G) = \frac{P(D \cap G)}{P(G)} = \frac{\frac{3}{10} \cdot \frac{1}{2}}{\frac{3}{10} \cdot \frac{1}{2} + \frac{7}{10} \cdot \frac{1}{10}} = \frac{3/20}{22/100} = \frac{15}{22}$

$\simeq 68{,}2\,\%$ (4)

(d) X : gain

$P(G) = \frac{22}{100}$ cf (c)

$P(\bar{G}) = 1 - P(G) = \frac{78}{100}$

ω	G	$\bar{G}$
P	22/100	78/100
X	+8	−2

$E(X) = 8 \cdot \frac{22}{100} - 2 \cdot \frac{78}{100} = \frac{20}{100} = 0{,}2.-$

(4)

/9

Ex 7

(a) $F(\vec{i}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$F(\vec{j}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\Rightarrow M_F = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$ (3)

(b) $F\begin{pmatrix} 2 \\ -1 \end{pmatrix} = M_F \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 2\cdot 2 + 2(-1) \\ 2\cdot 2 + 1(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (2)

(c) $\det M_F = 2\cdot 1 - 2\cdot 2 = -2$

$M_F^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$

$F^{-1}(\vec{a}) = M_F^{-1} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ (4)

[/22]

Ex 8

a) Soient $\vec{v}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{w}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R}^2$ et $\alpha, \beta \in \mathbb{R}$

a-t-on $F(\alpha\vec{v} + \beta\vec{w}) \overset{?}{=} \alpha F(\vec{v}) + \beta F(\vec{w})$

$\Leftrightarrow F\left(\alpha\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \beta\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) \overset{?}{=} \alpha F\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \beta F\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

$\Leftrightarrow F\begin{pmatrix} \alpha v_1 + \beta w_1 \\ \alpha v_2 + \beta w_2 \end{pmatrix} \overset{?}{=} \alpha\begin{pmatrix} v_1 + 2v_2 \\ v_2 \end{pmatrix} + \beta\begin{pmatrix} w_1 + 2w_2 \\ w_2 \end{pmatrix}$

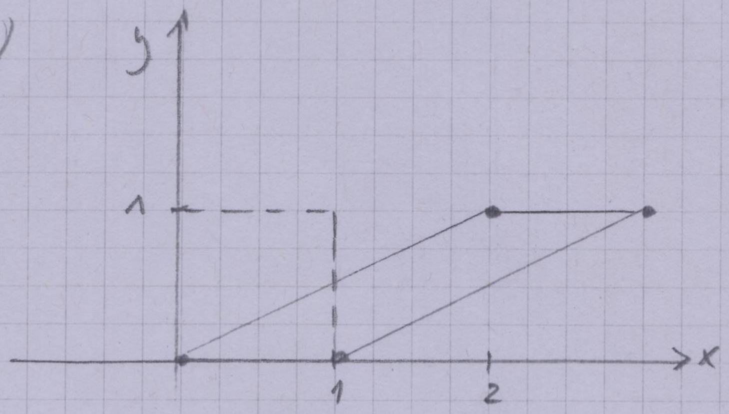
$\Leftrightarrow \begin{pmatrix} \alpha v_1 + \beta w_1 + 2(\alpha v_2 + \beta w_2) \\ \alpha v_2 + \beta w_2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} \alpha(v_1 + 2v_2) + \beta(w_1 + 2w \\ \alpha v_2 + \beta w_2 \end{pmatrix}$

$\Leftrightarrow \begin{pmatrix} \alpha v_1 + \beta w_1 + 2\alpha v_2 + 2\beta w_2 \\ \alpha v_2 + \beta w_2 \end{pmatrix} \overset{?}{=} \begin{pmatrix} \alpha v_1 + 2\alpha v_2 + \beta w_2 + 2\beta w_2 \\ \alpha v_2 + \beta w_2 \end{pmatrix}$

donc oui, F est linéaire (4)

c) $F(\vec{i}) = F\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$F(\vec{j}) = F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\Rightarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ c'est un cisaillement où $a = 2$ (3)

b)



$A\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ (3)

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

d) $A^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$ $A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
$= \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$ (1)

e) A inversible $\Leftrightarrow \det A \neq 0 \Leftrightarrow 1 \cdot 1 - 0 \cdot a \neq 0 \Leftrightarrow 1 \neq 0$
c'est toujours vrai, donc A inversible, $\forall a \in \mathbb{R}$ (2)

f) $A^{-1} = \dfrac{1}{\det A}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \dfrac{1}{1}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$ (1)

Ex 8 (suite)

g) La réciproque d'un cisaillement dont la matrice est A est une appl. linéaire dont la matrice est A^{-1}. Comme A^{-1} est de la forme $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ avec $k = -a$, c'est bien un cisaillement. (1)

h) La matrice de la composition de 2 appl. linéaires est le produit de leurs matrices.

soit $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ la matrice du 1er cisaillement

$B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ " " du 2e "

$A \cdot B = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$ qui est bien la matrice d'un cisaillement de paramètre $a+b$ (2)