Corrigé examen de maturité de mathématiques juin 2014

Q1

a) $B \cdot A = \begin{pmatrix} 2 & -3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 14 & 1 \end{pmatrix}$ 2

b) $B \cdot I^{2014} A = B \cdot I A = B A = \begin{pmatrix} 3 & 7 \\ 14 & 1 \end{pmatrix}$ 2

c) $A \cdot X = B \overset{\text{si } A^{-1} \text{ existe}}{\iff} \underbrace{A^{-1} \cdot A}_{I} \cdot X = A^{-1} \cdot B \iff X = A^{-1} \cdot B$ 1

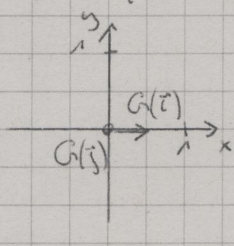
$\det(A) = 3 \cdot (-1) - 1 \cdot 2 = -5$ donc A^{-1} existe

$A^{-1} = \frac{1}{-5} \begin{pmatrix} -1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{pmatrix}$ 2

$X = \begin{pmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 8/5 & 7/5 \\ -7/5 & -18/5 \end{pmatrix}$ 1

Q2

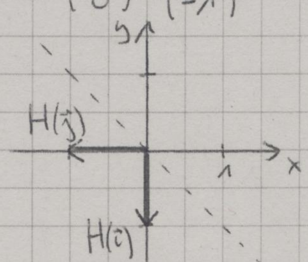
a) $G\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$ et $G\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 1



projection sur l'axe $y = 0$ 1

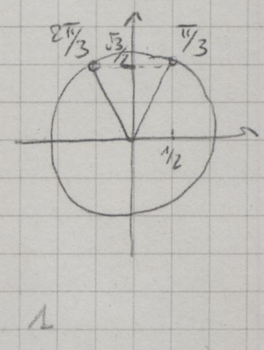
et réduction des distances horizontales d'un facteur 2 1

b) $H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ et $H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ 1



symétrie d'axe $y = -x$ 2

c) $M_F = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ 2



$F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = M_F \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1-\sqrt{3}}{2} \\ \frac{\sqrt{3}-1}{2} \end{pmatrix}$ 1

$$H\begin{pmatrix}1\\1\end{pmatrix} = M_H \cdot \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$ 1

d) $G\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \iff \begin{pmatrix}\frac{x}{2}\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \iff \begin{cases}\frac{x}{2} = 2\\0 = 2\end{cases}$ pas de préimage par G 2

$M_H \cdot \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \iff \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \iff \begin{cases}-y = 2\\-x = 2\end{cases}$

$\iff \begin{cases}x = -2\\y = -2\end{cases}$ la préimage de $\begin{pmatrix}2\\2\end{pmatrix}$ par H est $\begin{pmatrix}-2\\-2\end{pmatrix}$

e) $M_{(G \circ H)} = M_G \cdot M_H = \begin{pmatrix}1/2 & 0\\0 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1/2\\0 & 0\end{pmatrix}$ 2

f) ◦ $\det(M_F) = \frac{1}{4} + \frac{3}{4} = 1$ donc M_F est inversible

$M_{F^{-1}} = M_F^{-1} = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$ 2

◦ $\det(M_G) = 0$ donc M_G n'est pas inversible

$\Rightarrow M_{G^{-1}}$ n'existe pas 2

◦ $\det(M_H) = 0 - 1 = -1$ donc M_H est inversible

$M_{H^{-1}} = M_H^{-1} = \frac{1}{-1}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} = M_H$ 2

Q3

a1) $F(x) = \frac{1}{3}\ln(|x^3 - 5|)$ 3 a2) $F(x) = -\frac{1}{3}\cdot\frac{1}{x^3-5}$ 3

b) $\int_1^2 x \cdot e^{4-x^2}\,dx = -\frac{1}{2}e^{4-x^2}\Big|_1^2 = -\frac{1}{2}(e^0 - e^3) = \frac{1}{2}(e^3 - 1)$ 2

c) $F(x) = -\frac{1}{\sin(x)} + \frac{1}{3}\sin(3x) + C$ 2

$F\left(\frac{\pi}{6}\right) = -\frac{1}{\frac{1}{2}} + \frac{1}{3}\cdot 1 + C = -2 + \frac{1}{3} + C = 1$

$\Rightarrow C = \frac{8}{3}$ 2

D'où $F(x) = -\frac{1}{\sin(x)} + \frac{1}{3}\sin(3x) + \frac{8}{3}$

Q4

a) $V = \pi \int_0^6 \left(\frac{1}{4}\sqrt{x\cdot(36-x^2)}\right)^2 dx = \pi \cdot \int_0^6 \frac{1}{16} x\cdot(36-x^2)\ dx$

$= \pi \cdot \int_0^6 \frac{9}{4}x - \frac{1}{16}x^3\ dx = \pi \cdot \left[\frac{9}{8}x^2 - \frac{1}{64}x^4\right]\Big|_0^6$

$= \pi \cdot \left[\left(\frac{81}{2} - \frac{81}{4}\right) - 0\right] = \pi \cdot \frac{81}{4} \simeq 63{,}6\ cm^3$

b) $f'(x) = \frac{1}{4}\cdot\frac{1}{2}\left(x(36-x^2)\right)^{-1/2}\cdot(36-3x^2) = \frac{3\cdot(12-x^2)}{8\cdot\sqrt{x(36-x^2)}}$

$f'(x) = 0 \iff 3(12-x^2) = 0 \iff x = \pm\sqrt{12}$ donc $x = \sqrt{12}$

$f(\sqrt{12}) = \frac{1}{4}\sqrt{\sqrt{12}\cdot(36-12)} = \frac{1}{4}\sqrt{48\sqrt{3}} = \sqrt{3\sqrt{3}} \simeq 2{,}28$

Q5

a) $\left.\begin{array}{l} f(-2) = 7 + \frac{-6}{2} = 4 \\ g(-2) = -1 + 5 = 4 \end{array}\right\} \Rightarrow f(-2) = g(-2)$ $\quad \left.\begin{array}{l} f(2) = 7 + \frac{-2}{2} = 6 \\ g(2) = 1 + 5 = 6 \end{array}\right\} \Rightarrow f(2) = g$

b) $\left.\begin{array}{l} g(0) = 0 + 5 = 5 \\ h(0) = 0 - 0 + 5 = 5 \end{array}\right\} \Rightarrow g(0) = h(0)$ $\quad \left.\begin{array}{l} g(12) = 6 + 5 = 11 \\ h(12) = \frac{144}{5} - 22{,}8 + 5 = 11 \end{array}\right\} \Rightarrow g(1 = h(12$

c) $A = \int_{-2}^{2} (f(x) - g(x))\ dx + \int_0^{12} (g(x) - h(x))\ dx$ 4

$= \int_{-2}^{2} \left(7 + \frac{x}{2} - \frac{x^2}{2}\right) - \left(\frac{x}{2} + 5\right)\ dx + \int_0^{12} \left(\frac{x}{2} + 5\right) - \left(\frac{x^2}{5} - 1{,}9x + 5\right)\ dx$

$= \int_{-2}^{2} -\frac{x^2}{2} + 2\ dx + \int_0^{12} -\frac{x^2}{5} + 2{,}4x\ dx$

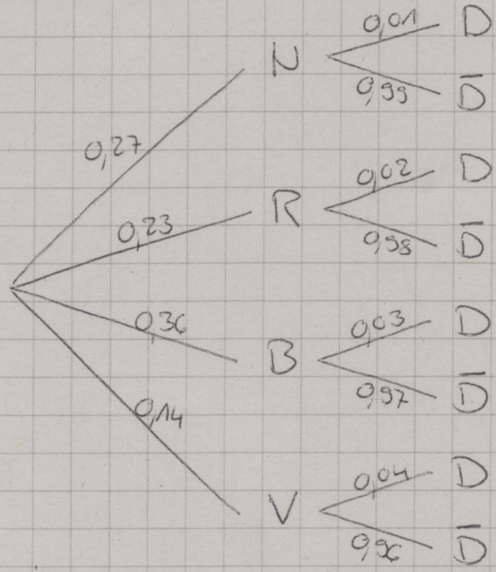
$= -\frac{1}{6}x^3 + 2x \Big|_{-2}^{2} + \left(-\frac{1}{15}x^3 + \frac{6}{5}x^2\right)\Big|_0^{12}$ 2

$= \left(-\frac{8}{6} + 4\right) - \left(\frac{8}{6} - 4\right) + \left(-\frac{144\cdot 4}{5} + \frac{6\cdot 144}{5}\right) - 0$

$= -\frac{8}{3} + 8 + \frac{288}{5} = \frac{-40 + 120 + 864}{15} = \frac{944}{15} \simeq 62{,}93\ du$

2

Q6



N : "stylo noir"

R : "stylo rouge"

B : "stylo bleu"

V : "stylo vert"

D : "défectueux"

$\bar{D}$: "en bon état"

a) $p(R \cap \bar{D}) = 0{,}23 \cdot 0{,}98 = 0{,}2254$ 3

b) $p(D) = p(N \cap D) + p(R \cap D) + p(B \cap D) + p(V \cap D)$

$= 0{,}27 \cdot 0{,}01 + 0{,}23 \cdot 0{,}02 + 0{,}36 \cdot 0{,}03 + 0{,}14 \cdot 0{,}04$

$= 0{,}0237$ 4

c) $p(V/D) = \dfrac{p(V \cap D)}{p(D)} = \dfrac{0{,}14 \cdot 0{,}04}{0{,}0237} \simeq 0{,}2363$ 4

Q7

a1) $p = 0{,}15 \quad n = 18 \quad k = 10$

$p(10 \text{ succès}) = C_{10}^{18} \cdot (0{,}15)^{10} \cdot (0{,}85)^{18-10} \simeq 6{,}876 \cdot 10^{-5} \simeq 0{,}0069\%$

4

a2) $p = 0{,}15 \quad n = 18 \quad k \geqslant 2$

$p(\text{au moins 2 succès}) = 1 - p(0 \text{ succès}) - p(1 \text{ succès})$ 2

$= 1 - C_0^{18} \, (0{,}15)^0 \cdot (0{,}85)^{18} - C_1^{18} \cdot (0{,}15)^1 \cdot (0{,}85$

$\simeq 1 - 0{,}0536 - 0{,}1704$

$\simeq 0{,}7759$ 2

b) $p = 0{,}15 \quad n = ? \quad k \geqslant 1$

$p(\text{au moins 1 succès}) = 1 - p(0 \text{ succès})$

$= 1 - \underbrace{C_0^n \cdot (0{,}15)^0}_{=1} (0{,}85)^n = 1 - (0{,}85)^n$

2

$$1-(0{,}85)^n \geqslant 0{,}99 \quad 1$$

$$(0{,}85)^n \leq 0{,}01 \quad 1$$

$$\log(0{,}85)^n \leq \log(0{,}01)$$

$$n \cdot \log(0{,}85) \leq \log(0{,}01) \quad 1$$

$$n \geqslant \frac{\log(0{,}01)}{\log(0{,}85)} \simeq 28{,}3$$

Il faut donc au minimum 29 Outardes des Zavanes. 1

Q8

a) $A^2 = \begin{pmatrix} a^2+bc & 0 \\ 0 & bc+a^2 \end{pmatrix}$ Si $A^2 = 0$ alors $a^2+bc=0$

$I-A = \begin{pmatrix} 1-a & -b \\ -c & 1+a \end{pmatrix}$, $\det(I-A) = 1-a^2-bc = 1-\underbrace{(a^2+bc)}_{=0} = 1$

$\det(I-A) \neq 0$ donc $(I-A)$ est inversible. Donc Vrai.

b) F linéaire $\Leftrightarrow \begin{cases} F(\vec{u}+\vec{v}) = F(\vec{u}) + F(\vec{v}) \\ F(\alpha\vec{u}) = \alpha F(\vec{u}) \end{cases}$ $\forall \vec{u}, \vec{v} \in \mathbb{R}^2, \forall \alpha \in$

idem pour G ...

- $H(\vec{u}+\vec{v}) \overset{\text{déf } H}{=} F(\vec{u}+\vec{v}) + G(\vec{u}+\vec{v}) \overset{F \text{ et } G \text{ linéaires}}{=} F(\vec{u}) + F(\vec{v}) + G(\vec{u}) + G(\vec{v})$
  $= F(\vec{u}) + G(\vec{u}) + F(\vec{v}) + G(\vec{v}) \overset{\text{déf } H}{=} H(\vec{u}) + H(\vec{v})$

- $H(\alpha\vec{u}) \overset{\text{déf } H}{=} F(\alpha\vec{u}) + G(\alpha\vec{u}) \overset{F \text{ et } G \text{ linéaires}}{=} \alpha F(\vec{u}) + \alpha G(\vec{u}) = \alpha\big(F(\vec{u}) + G(\vec{u})\big)$
  $\overset{\text{déf } H}{=} \alpha \cdot H(\vec{u})$

Donc Vrai.

c) Faux contre-exemple : on choisit $p = 0{,}1$

- $B(2;4;p) = C_2^4 \cdot (0{,}1)^2 \cdot (1-0{,}1)^{4-2} = 0{,}0486$
- $B(5;10;p) = C_5^{10} \cdot (0{,}1)^5 \cdot (1-0{,}1)^{10-5} \simeq 0{,}0015$