Corrigé des exercices du chapitre 7

1

a. $(f+g)(x) = f(x)+g(x) = x^3+2x^2+3x^2-1 = x^3+5x^2-1$

$(f-g)(x) = f(x)-g(x) = x^3+2x^2-(3x^2-1) = x^3-x^2+1$

$(f\cdot g)(x) = f(x)\cdot g(x) = (x^3+2x^2)(3x^2-1) = 3x^5+6x^4-x^3-2x^2$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^3+2x^2}{3x^2-1}$

$D_{f+g} = D_{f-g} = D_{f\cdot g} = D_f \cap D_g = \mathbb{R}$

$D_{f/g} = D_f \cap D_g \setminus \{x \mid g(x)=0\} = \mathbb{R} \setminus \{\pm \frac{\sqrt{3}}{3}\}$

b. $(f+g)(x) = f(x)+g(x) = \sqrt{1+x} + \sqrt{1-x}$

$(f-g)(x) = f(x)-g(x) = \sqrt{1+x} - \sqrt{1-x}$

$(f\cdot g)(x) = f(x)\cdot g(x) = \sqrt{1+x}\cdot\sqrt{1-x} = \sqrt{1-x^2}$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \sqrt{\frac{1+x}{1-x}}$

$D_{f+g} = D_{f-g} = D_{f\cdot g} = [-1;+\infty[\cap]-\infty;1] = [-1;1]$

$D_{f/g} = [-1,1] \setminus \{1\} = [-1;1[$

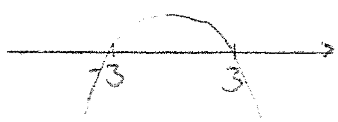
c. $(f+g)(x) = f(x)+g(x) = \sqrt{9-x^2} + \sqrt{x^2-1}$

$(f-g)(x) = f(x)-g(x) = \sqrt{9-x^2} - \sqrt{x^2-1}$

$(f\cdot g)(x) = f(x)\cdot g(x) = \sqrt{9-x^2}\cdot\sqrt{x^2-1} = \sqrt{-x^4+10x^2-9}$

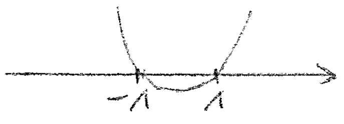
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}} = \sqrt{\frac{9-x^2}{x^2-1}}$

D_f: $9-x^2 \geq 0 \iff (3-x)(3+x) \geq 0$



$D_f = [-3;3]$

D_g: $x^2-1 \geq 0 \iff (x-1)(x+1) \geq 0$



$D_g =]-\infty;-1] \cup [1;+\infty[$

$D_{f+g} = D_{f-g} = D_{f\cdot g} = D_f \cap D_g = [-3;-1] \cup [1;3]$

$D_{f/g} = [-3;-1[\cup]1;3]$

2

a. $(f\circ g)(x) = f(g(x)) = f(x^2-3x+5) = 2(x^2-3x+5)-5 = 2x^2-6x+5$

$(g\circ f)(x) = g(f(x)) = g(2x-5) = (2x-5)^2-3(2x-5)+5 = 4x^2-26x+45$

$(g\circ g)(x) = g(g(x)) = g(x^2-3x+5) = (x^2-3x+5)^2-3(x^2-3x+5)+5$

$= x^4-6x^3+19x^2-30x+25-3x^2+9x-15+5 = x^4-6x^3+16x^2-21x+15$

b. $(f\circ g)(x) = f(g(x)) = f(x^2-2) = \dfrac{2(x^2-2)-1}{(x^2-2)+4} = \dfrac{2x^2-5}{x^2+2}$

$(g\circ f)(x) = g(f(x)) = g\left(\dfrac{2x-1}{x+4}\right) = \left(\dfrac{2x-1}{x+4}\right)^2-2 = \dfrac{4x^2-4x+1-2(x^2+8x+16)}{x^2+8x+16} = \dfrac{2x^2-20x-31}{x^2+8x+16}$

$(g\circ g)(x) = g(g(x)) = g(x^2-2) = (x^2-2)^2-2 = x^4-4x^2+2$

c. $(f\circ g)(x) = f(g(x)) = f\left(\dfrac{2x-1}{x-3}\right) = \dfrac{3\left(\frac{2x-1}{x-3}\right)-4}{\left(\frac{2x-1}{x-3}\right)+1} = \dfrac{\frac{6x-3-4(x-3)}{x-3}}{\frac{2x-1+x-3}{x-3}} = \dfrac{\frac{2x+9}{x-3}}{\frac{3x-4}{x-3}} = \dfrac{2x+9}{3x-4}$

$(g\circ f)(x) = g(f(x)) = g\left(\dfrac{3x-4}{x+1}\right) = \dfrac{2\left(\frac{3x-4}{x+1}\right)-1}{\frac{3x-4}{x+1}-3} = \dfrac{\frac{6x-8-(x+1)}{x+1}}{\frac{3x-4-3(x+1)}{x+1}} = \dfrac{\frac{5x-9}{x+1}}{\frac{-7}{x+1}} = -\dfrac{5x-9}{7}$

$(g\circ g)(x) = g(g(x)) = g\left(\dfrac{2x-1}{x-3}\right) = \dfrac{2\left(\frac{2x-1}{x-3}\right)-1}{\frac{2x-1}{x-3}-3} = \dfrac{\frac{4x-2-(x-3)}{x-3}}{\frac{2x-1-3(x-3)}{x-3}} = \dfrac{\frac{3x+1}{x-3}}{\frac{-x+8}{x-3}} = \dfrac{3x+1}{8-x}$

d. $(f\circ g)(x) = f(g(x)) = f(\sqrt{1-x}) = \sqrt{(\sqrt{1-x})^2-1} = \sqrt{1-x-1} = \sqrt{-x}$

$(g\circ f)(x) = g(f(x)) = g(\sqrt{x^2-1}) = \sqrt{1-\sqrt{x^2-1}}$

$(g\circ g)(x) = g(g(x)) = g(\sqrt{1-x}) = \sqrt{1-\sqrt{1-x}}$

3

a. $f_1(x) = \dfrac{x-1}{x+1}$

$f_2(x) = (f\circ f)(x) = \dfrac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \dfrac{\frac{x-1-x-1}{x+1}}{\frac{x-1+x+1}{x+1}} = \dfrac{\frac{-2}{x+1}}{\frac{2x}{x+1}} = \dfrac{-2}{2x} = -\dfrac{1}{x}$

$f_3(x) = (f\circ f_2)(x) = f\left(-\dfrac{1}{x}\right) = \dfrac{-\frac{1}{x}-1}{-\frac{1}{x}+1} = \dfrac{\frac{-1-x}{x}}{\frac{-1+x}{x}} = \dfrac{-1-x}{-1+x} = -\dfrac{x+1}{x-1}$

$f_4(x) = (f\circ f_3)(x) = f\left(-\dfrac{x+1}{x-1}\right) = \dfrac{-\frac{x+1}{x-1}-1}{-\frac{x+1}{x-1}+1} = \dfrac{\frac{-x-1-x+1}{x-1}}{\frac{-x-1+x-1}{x-1}} = \dfrac{-2x}{-2} = x$

$f_5(x) = (f\circ f_4)(x) = f(x) = \dfrac{x-1}{x+1} = f_1(x)$

b. $f_6 = f_2$; $f_7 = f_3$; $f_8 = f_4$; … $f_{1000} = f_4$ $\quad f_{1000}(3) = 3$

c. $(f\circ g)(x) = f\left(\frac{x+1}{1-x}\right) = \dfrac{\frac{x+1}{1-x}-1}{\frac{x+1}{1-x}+1} = \dfrac{\frac{x+1-1+x}{1-x}}{\frac{x+1+1-x}{1-x}} = \frac{2x}{2} = x$.

$(g\circ f)(x) = g\left(\frac{x-1}{x+1}\right) = \dfrac{\frac{x-1}{x+1}+1}{1-\frac{x-1}{x+1}} = \dfrac{\frac{x-1+x+1}{x+1}}{\frac{x+1-x+1}{x+1}} = \frac{2x}{2} = x$

Donc g est la réciproque de f.

$\boxed{4.}$

$f(x) = \frac{1}{1-x}$ $\qquad x \xrightarrow[v]{} 1-x \xrightarrow[u]{} \frac{1}{1-x}$ $\qquad f = u\circ v$

$g(x) = 1-2x$ $\qquad x \xrightarrow[w]{} 2x \xrightarrow[v]{} 1-2x$ $\qquad g = v\circ w$

$h(x) = \frac{x-1}{x}$ $\qquad x \xrightarrow[u]{} \frac{1}{x} \xrightarrow[v]{} 1-\frac{1}{x}$ $\qquad h = v\circ u$

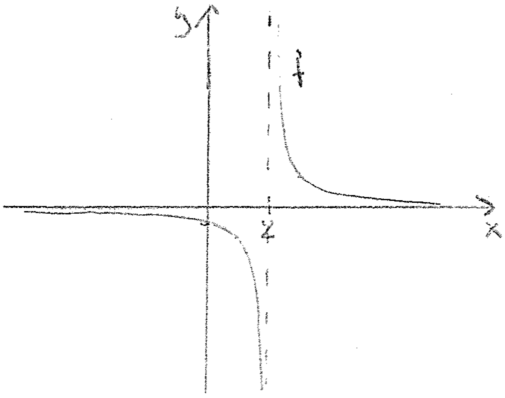
$i(x) = x$ $\qquad x \xrightarrow[u]{} \frac{1}{x} \xrightarrow[u]{} \frac{1}{\frac{1}{x}} = x$ $\qquad i = u\circ u$

$k(x) = \frac{x}{x-2}$ $\qquad x \xrightarrow[u]{} \frac{1}{x} \xrightarrow[w]{} \frac{2}{x} \xrightarrow[v]{} 1-\frac{2}{x} = \frac{x-2}{x} \xrightarrow[u]{} \frac{1}{\frac{x-2}{x}} = \frac{x}{x-}$

$k = u\circ v\circ w\circ u$

$\boxed{7.}$

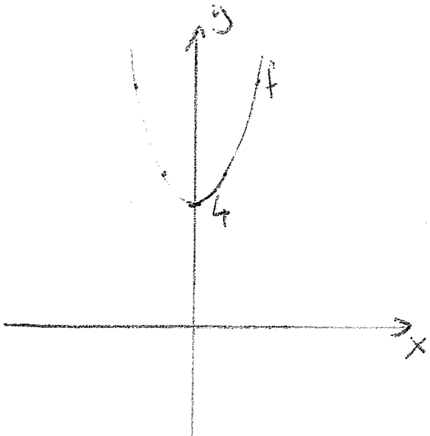
a. $f(x) = \frac{1}{x-2}$



f n'est pas bijective de $\mathbb{R}$ dans $\mathbb{R}$

f est bijective de $\mathbb{R}\setminus\{2\}$ dans $\mathbb{R}\setminus\{0\}$

b. $f(x) = x^2+4$

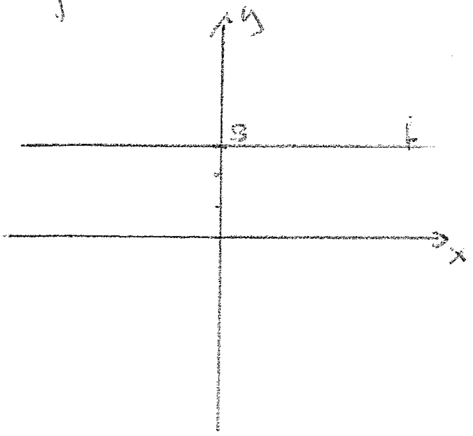


f n'est pas bijective de $\mathbb{R}$ dans $\mathbb{R}$

f est bijective de $\mathbb{R}_+$ dans $[4;+\infty[$

" " " " $\mathbb{R}_-$ dans $[4;+\infty[$

c. $f(x) = 3$



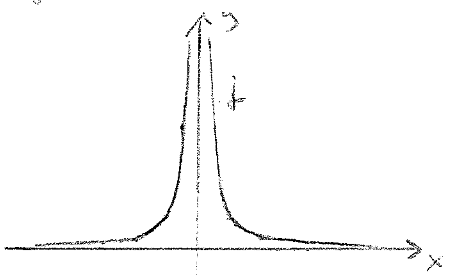
f n'est pas bijective de $\mathbb{R}$ dans $\mathbb{R}$

f est bijective de $\{0\}$ dans $\{3\}$

" " " de $\{1\}$ dans $\{3\}$

" " " de $\{a\}$ dans $\{3\}$, $a \in \mathbb{R}$

d. $f(x) = \frac{1}{x^2}$



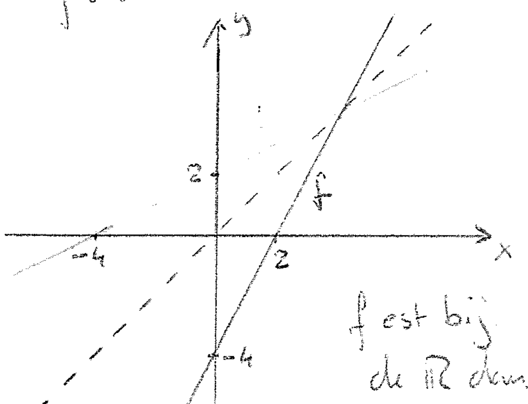
f n'est pas bijective de $\mathbb{R}$ dans $\mathbb{R}$

f est bijective de $\mathbb{R}_+^*$ dans $\mathbb{R}_+^*$

" " " " $\mathbb{R}_-^*$ dans $\mathbb{R}_+^*$

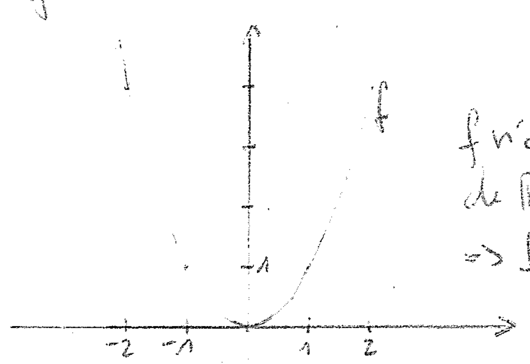
$\boxed{8}$

a. $f(x) = 2x - 4$ de $\mathbb{R}$ dans $\mathbb{R}$



f est bij de $\mathbb{R}$ dans $\mathbb{R}$

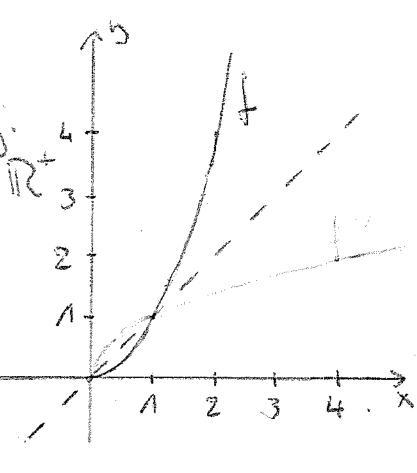
b. $f(x) = x^2$ de $\mathbb{R}$ dans $\mathbb{R}$



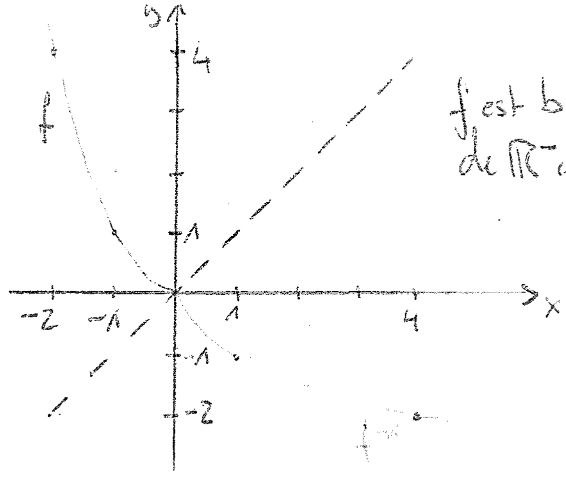
f n'est pas bij. de $\mathbb{R}$ dans $\mathbb{R}$ $\Rightarrow$ f n'admet pas de réciproque

c. $f(x) = x^2$ de $\mathbb{R}^+$ dans $\mathbb{R}^+$

f est bij. de $\mathbb{R}^+$ ds $\mathbb{R}^+$



d. $f(x) = x^2$ de $\mathbb{R}^-$ dans $\mathbb{R}^+$



f est bij. de $\mathbb{R}^-$ ds $\mathbb{R}^+$

9

a. $f(x) = 3x+5$

(1) $D_f = \mathbb{R}$

(2) $y = f(x) \iff y = 3x+5 \iff y-5 = 3x \iff \frac{y-5}{3} = x$

$f^{-1}(y) = x \Rightarrow f^{-1}(y) = \frac{y-5}{3}$

(3) On remplace y par x. $f^{-1}(x) = \frac{x-5}{3}$

(4) $f : \mathbb{R} \longrightarrow \mathbb{R}$ bij. a pour réciproque : $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ bij.
$x \longmapsto 3x+5$ $\qquad$ $x \longmapsto \frac{x-5}{3}$

b. $f(x) = \frac{1}{3x-2}$

(1) $D_f = \mathbb{R} \setminus \{\frac{2}{3}\}$

(2) $y = f(x) \iff y = \frac{1}{3x-2} \iff (3x-2)y = 1 \iff 3xy - 2y = 1$

$\iff 3xy = 2y+1 \iff x = \frac{2y+1}{3y}$ $\Rightarrow$ condition : $(y \neq 0$

$f^{-1}(y) = x \Rightarrow f^{-1}(y) = \frac{2y+1}{3y}$

(3) On remplace y par x : $f^{-1}(x) = \frac{2x+1}{3x}$

(4) $f : \mathbb{R} \setminus \{\frac{2}{3}\} \longrightarrow \mathbb{R} \setminus \{0\}$ bij.
$x \longmapsto \frac{1}{3x-2}$

a pour réciproque :

$f^{-1} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{\frac{2}{3}\}$ bij.
$x \longmapsto \frac{2x+1}{3x}$

c. $f(x) = 2 - 3x^2$

(1) $D_f = \mathbb{R}$

(2) $y = f(x) \iff y = 2-3x^2 \iff 3x^2 = 2-y \iff x^2 = \frac{2-y}{3}$

$\iff x = \pm\sqrt{\frac{2-y}{3}}$ il ne peut pas y avoir 2 préimages

on choisit l'une des 2 : $x = \sqrt{\frac{2-y}{3}} \Rightarrow f^{-1}(y) = \sqrt{\frac{2-y}{3}} \Rightarrow x \geq 0$

condition : $\frac{2-y}{3} \geq 0 \iff 2-y \geq 0 \iff 2 \geq y$

(3) On remplace y par x : $f^{-1}(x) = \sqrt{\frac{2-x}{3}}$

(4) $f : [0;+\infty[\longrightarrow]-\infty;2]$ bij.
$x \longmapsto 2-3x^2$

a pour réciproque $f^{-1} :]-\infty;2] \longrightarrow [0;+\infty[$ bij.
$x \longmapsto \sqrt{\frac{2-x}{\ldots}}$

d. $f(x) = \dfrac{3x+2}{2x-5}$

(1) $D_f = \mathbb{R} \setminus \{\frac{5}{2}\}$

(2) $y = f(x) \iff y = \dfrac{3x+2}{2x-5} \iff (2x-5)y = 3x+2$

$\iff 2xy - 5y = 3x + 2 \iff 2xy - 3x = 5y + 2$

$\iff x(2y-3) = 5y+2 \iff x = \dfrac{5y+2}{2y-3}$ cond. $2y - 3 \neq 0$

$\iff y \neq \frac{3}{2}$

$f^{-1}(y) = x \implies f^{-1}(y) = \dfrac{5y+2}{2y-3}$

(3) On remplace y par x : $f^{-1}(x) = \dfrac{5x+2}{2x-3}$

(4) $f : \mathbb{R} \setminus \{\frac{5}{2}\} \longrightarrow \mathbb{R} \setminus \{\frac{3}{2}\}$ bij.

$x \longmapsto \dfrac{3x+2}{2x-5}$

a pour réciproque :

$f^{-1} : \mathbb{R} \setminus \{\frac{3}{2}\} \longrightarrow \mathbb{R} \setminus \{\frac{5}{2}\}$ bij.

$x \longmapsto \dfrac{5x+2}{2x-3}$

e. $f(x) = \sqrt{3-x}$

(1) $3 - x \geqslant 0 \iff 3 \geqslant x \implies D_f =]-\infty, 3]$

(2) $y = f(x) \iff y = \sqrt{3-x}$ $(y \geqslant 0$

$\iff y^2 = 3 - x \iff x = 3 - y^2$

$f^{-1}(y) = x \implies f^{-1}(y) = 3 - y^2$

(3) $f^{-1}(x) = 3 - x^2$

(4) $f :]-\infty, 3] \longrightarrow [0, +\infty[$ bij.

$x \longmapsto \sqrt{3-x}$

a pour réciproque :

$f^{-1} : [0; +\infty[\longrightarrow]-\infty; 3]$ bij.

$x \longmapsto 3 - x^2$

f. $f(x) = x^2 - 4x + 2$

(1) $D_f = \mathbb{R}$

(2) $y = f(x) \iff y = x^2 - 4x + 2 \iff x^2 - 4x + (2-y) = 0$

$\Delta = 16 - 4(2-y) = 8 + 4y \quad x_{1,2} = \dfrac{4 \pm \sqrt{8+4y}}{2} = 2 \pm \sqrt{y+2}$ $\begin{cases} 2 + \sqrt{y+2} \\ 2 - \sqrt{y+2} \end{cases}$

on choisit un des 2 solutions : $x = 2 + \underbrace{\sqrt{y+2}}_{\geqslant 0} \implies (x \geqslant 2$

condition : $y + 2 \geqslant 0 \iff y \geqslant -2$

$f^{-1}(y) = x \quad \Rightarrow \quad f^{-1}(y) = 2 + \sqrt{y+2}$

(3) $f^{-1}(x) = 2 + \sqrt{x+2}$

(4) $f : [2, +\infty[\longrightarrow [-2, +\infty[$ bij.

$\quad x \longmapsto x^2 - 4x + 2$

$f^{-1} : [-2, +\infty[\longrightarrow [2, +\infty[$ bij.

$\quad x \longmapsto 2 + \sqrt{x+2}$